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STUDY OF QUASI-OPTIMUM FEEDBACK CONTROL TECHNIQUES

by Bernard Friedland, Frederick E. Thau, Victor D. Cohen, and Jordan Ellis

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PREFACE

This report contains the results of a "Study of Quasi-Optimum Feedback Control Techniques", performed during 1965 at the General Precision Aerospace Research Center, under Contract NAS 2-2648 with the Ames Research Center, National Aeronautics and Space Administration.

The principal investigator was Dr. Bernard Friedland; contributors included Dr. Frederick E. Thau and Messrs. Victor D. Cohen and Jordan Ellis.

CONTENTS

INTRODUCTION AND SUMMARY

A major limitation to the use of modern variational control theory for the design of practical feedback control systems is the need to solve a two-point boundary-value problem of ordinary differential equations in real-time. In most situations, the realization of the exact solution to the two-point boundary-value problem is not feasible in view of the cost and size of equipment which such a computation would entail. Moreover, the optimum performance, in many instances, is only negligibly superior to that obtainable with a simpler, non-optimum control computation. For this reason most practical feedback control systems continue to be designed by conventional frequency-domain or cut-and-try techniques. There is ample evidence, however, of the shortcomings of conventional techniques for the design of control systems for complex processes; a clear need exists for design techniques which employ the modern variational approach but do not entail the solution of a complex two-point boundary-value problem. This need motivated the quasi-optimum control technique of this study.

The basis of the technique is the observation that a complex process can often be approximated by a much simpler process for which the exact optimum feedback control law can be expressed in closed-form. This control law, however, may not be adequate for the actual process and must be corrected to account for the difference between the actual process and its simplified model. For this approach to be practical; it is necessary that the required correction be computed without prior knowledge of the exact control law for the actual process.

It is shown in Part 1 of this report that if an exact solution to the "simplified problem" can be found, then the required quasi-optimum control law can indeed be computed.

In principle, the optimum control law for the exact process can be expressed by the relation

$$u^* = \sigma(x, p(x))$$

where u^* is the optimum control, x is the state of the process, $\sigma(\cdot)$ is a nonlinear transformation chosen to maximize the "Hamiltonian" of the process, and $p(x)$ is the "adjoint" or "costate" vector which must be computed as the solution to the two-point boundary-value problem. The nonlinear transformation σ is generally easy to determine; the main difficulty lies in computing $p(x)$. In the technique investigated here, the quasi-optimum control law is expressed as

$$u = \sigma(x\,,\,P(X) + M(X)\xi)\;, \qquad \xi = x - X$$

where X is the state vector of the simplified process and $P(X)$ is the corresponding adjoint vector. (By assumption, $P(X)$ can be computed from X.) The "correction matrix" M can be computed by means of the matrix Riccati equation

$$-\dot{M} = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX}$$

where the coefficient matrices $H_{XP}, \ldots, H_{XX}$, are matrices of second partial derivatives of the Hamiltonian of the exact problem evaluated at $x = X$. A set of "auxiliary equations"

$$\dot{\xi} = H_{XP}\xi + H_{PP}\psi$$

$$\dot{\psi} = -H_{XX}\xi - H_{PX}\psi$$

where $\psi = M\xi$

can also be used to determine M. Boundary conditions for the Riccati equations and the auxiliary equations are obtained by linearizing the boundary and transversality conditions for the actual process.

This approach is similar in some respects to a technique described by Pearson [1]. Pearson's technique, however, is limited to quadratic performance criteria, and to processes in which no "hard" control variable constraints are present. These limitations do not apply in our technique. There is also a close connection between our technique and the so-called "second-variation" techniques [2, 3, 4, 5]. The principal distinctions between our technique and the second-variation techniques are: (1) in our technique the "linearization" is performed with respect to a simplified process rather than a "nominal trajectory" of the exact process; (2) in our technique the correction is made to the adjoint vector rather than directly to the control variable. The latter distinction is particularly significant, as it permits treatment of hard control constraints which cannot be treated by the earlier second-variation techniques. The use of our approach for linearizing about a nominal trajectory is described in Section 1.5, but not considered in detail. A discussion of the use of the technique for "mildly-nonlinear" processes is given in Section 1.6.

The underlying assumption of the technique is that the difference ξ between the state x of the actual process and the state X of the simplified model is small. A theoretical investigation of the general relationship between the magnitude of ξ and the performance of the quasi-optimum control law was deferred for future study. Instead, the validity of the technique was illustrated by means of several practical application studies, the results of which are described in Part 2 of this report.

The first example considered was time-optimum control of the linear process $\ddot{x} + a\dot{x} = u$ subject to the constraint $|u| \leq 1$. This form of "Bushaw's problem" was selected to verify the applicability of the technique to a problem with a "hard" control constraint and to compare the quasi-optimum solution with the well-known exact solution. In applying the technique, the damping coefficient a was treated as a state variable, which, in the simplified problem, was assumed to be zero. The switching curve obtained by use of the quasi-optimum technique differed from the exact switching curve by only a few percent for a substantial range of x and $\dot{x}$ when $a = 0.3$ (which is not really small). The switching curve for the simplified process, on the other hand, was quite far from the exact curve. From this example it would appear that the technique is capable of giving good results even when ξ is not negligible.

The second application considered was minimum-time constant-thrust rendezvous in free space. A target-referenced polar coordinate system was used to describe the relative motion. The simplified process was obtained by assuming no relative tangential velocity; the resulting control law is the well-known one for the process $\ddot{x} = u$ with $|u| = 1$, but this control law was completely inadequate if the initial tangential velocity was nonzero. By use of the quasi-optimum control technique, however, satisfactory performance was achieved even for problems in which the initial velocity was only tangential, or when the tangential and radial components of velocity had equal magnitudes. Here again good results were achieved for fairly large values of ξ.

The third example treated was flight control of a flexible booster. A quadratic performance criterion, consisting of a weighted sum of the drift of the vehicle from the trajectory plane at burnout and the integral of the square of the bending moment was selected. The simplified model was obtained by assuming the vehicle to be a rigid body, and explicit, closed-form expressions for the gains were obtained. The performance of this control law was found to be inadequate in the presence of any appreciable flexibility. When the quasi-optimum control law was used, however, good performance was achieved for moderately flexible vehicles. When the flexibility was increased beyond a certain point, then even the quasi-optimum control proved inadequate.

The application of the technique to minimum-time, three-axis attitude control of a space vehicle with small but not negligible gyroscopic cross-axis coupling was considered. For the simplified model the gyroscopic couplings were assumed to be zero; consequently the simplified controller comprises three independent single-axis controls. The computations for the cross-axis couplings, which entail only simple but tedious algebraic manipulations, are not complete.

Another example considered was guidance of a maneuverable reentry vehicle. This study was started late in the year and substantial effort will be required to complete the study.

The combination of the quasi-optimum control technique with a statistical parameter estimation technique as a method of achieving adaptive control is described in Section 1.6; the validity of the technique remains to be established, however.

Some of the theory of Part I and the first illustrative example of Part 2 was presented at the 1965 Joint Automatic Control, Troy, New York, June 22-25, 1965 in a paper by B. Friedland entitled "A Technique of Quasi-Optimum Control." (Preprint Volume pp. 244-252). The general theory, including the alternate techniques discussed in Section 1.5 will be presented at the Third Congress of the International Federation of Automatic Control, (to be held in London, England, June 20-25, 1966) in a paper by B. Friedland and P. E. Sarachik, entitled "A Unified Approach to Suboptimum Control.

Part 1

THEORY

1.1 PROBLEM FORMULATION

We begin by stating the optimization problem in the manner of Pontryagin et al [6]; the process is described by the system of first-order differential equations

$$\dot{x} = f(x, u) \tag{1}$$

where $x = \{x_0, x_1, \ldots, x_n\}$ is the state vector, $u = \{u_1, u_2, \ldots, u_r\}$ is the control vector, and $f = \{f_0, f_1, f_1, \ldots, f_n\}$ is a vector-valued function. The component x_0 of x is a measure of the performance. A feedback control law $u = u(x)$ is to be determined which takes the process from some current * state $x(t)$ to a final state $x(T)$, such that the performance index $x_0(T)$ is a minimum, and the remaining n states satisfy the boundary conditions

$$\varphi\,(x(T)) = 0 \tag{2}$$

where $\varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_s\}$, $s \leq n$. The terminal time T may be either free or specified. In addition, the control u may be required to be a member of a closed, bounded set Ω.

The structure of the optimum controller can be determined by the maximum principle of Pontryagin [6]. Define the Hamiltonian function:

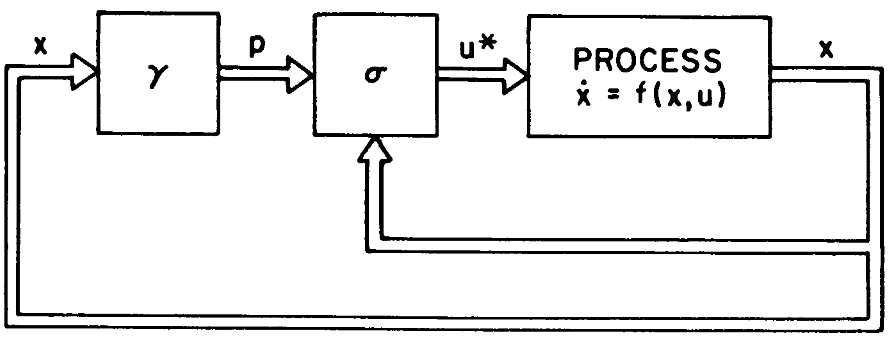
$$h(p, x, u) = p' \, f(x, u) \tag{3}$$

where $p = \{p_0, p_1, \ldots, p_n\}$ and (′) denotes transposition, and where p satisfies the adjoint equation

$$\dot{p} = -\mathrm{grad}_x \, h = -h_x \tag{4}$$

---

*The current time is denoted by the variable t, terminal time by T; time when it is used as an independent variable is denoted by τ, e.g., $t < \tau < T$.



STRUCTURE OF OPTIMUM CONTROL SYSTEM
FIGURE I

It is seen from (1) that

$$\dot{x} = \text{grad}_p\, h = h_p \tag{5}$$

Necessary conditions for the existence of an optimum control u^* are:

(i) h is maximum with respect to $u \in \Omega$, that is,

$$h(x, u^*, p) = \max_{u \in \Omega} h(x, u, p) \tag{6a}$$

(ii)

$$h(x, u^*, p) = \text{const} \tag{6b}$$

(iii) The adjoint vector satisfies the "transversality conditions"

$$p(T) = \left[\frac{-1}{\Phi' \lambda}\right] \tag{7}$$

where λ is a vector of s constants

and

$$\Phi = \left[\frac{\partial \varphi_i}{\partial x_j}\right] \qquad i = 1, 2, \ldots, s; \; j = 0, 1, \ldots, n$$

The optimum control system may thus be conceived as having the structure shown in Figure 1. The transformation σ of the process state vector x and the adjoint vector p into the control

$$u^* = \sigma(p, x) \tag{8}$$

is defined by (6a), and is determined by maximizing the Hamiltonian (3) with $u \in \Omega$. Equations (4), (5) and (8), together with boundary conditions (2) and (7), define a two-point boundary-value problem. Given the current state $x(t)$ (if a solution of the boundary-value problem exists), then the adjoint $p(t)$ may be determined as the solution to the two-point boundary-value problem. Thus, (2), (4), (5), (7) and (8) define a transformation γ of the current state $x(t)$ into the adjoint $p(t)$. For most applications, this transformation implicit in the solution of the two-point boundary-value problem cannot be obtained analytically. Consequently, it is desirable to develop "quasi-optimum" procedures which avoid solving the two-point boundary-value problem.

1.2 QUASI-OPTIMUM TECHNIQUE

The basis of the approach to be developed is the assumption that the original process can be approximated by a simpler process which has the following properties:

(A) The difference between the state of the original process and the simpler process is small.

(B) The optimum control law for the simplified system can be found exactly; that is, an explicit expression for the solution of the two-point boundary-value problem for the simplified process can be found.

Suppose the state x can be regarded as the sum of two terms

$$x = X + \xi \tag{9}$$

where X is the state of the simplified process. Then (1) can be written

$$\dot{X} + \dot{\xi} = f(X + \xi\ ,\ u)$$

Furthermore, assume that ξ is small (i.e., $\| \xi(\tau) \| < \epsilon$ for $t < \tau < T$ where ϵ is sufficiently small). Then the origianl system can be approximated by the system

$$\dot{X} = \lim_{\xi \to 0} f(X + \xi, u) = F(X , u) \tag{10}$$

where $\varphi(X(T)) = 0$.

By defining a Hamiltonian $H = P' F(X , u)$, a corresponding two-point boundary value for the simplified system can be derived, i.e.,

$$\begin{aligned} \dot{X} &= H_P \\ \dot{P} &= -H_X \end{aligned} \tag{11}$$

and

$$P(T) = \left[\frac{1}{\Phi' \Lambda} \right]$$

where Λ is an s-dimensional vector of "slack" variables.

The adjoint vector P, which by assumption (B) can be solved for in terms of X, may be regarded as an approximate solution for p of the exact problem.

As ξ increases, this approximation deteriorates, and may be inadequate. Consequently, it is desirable to include the effects of the state "error" ξ more exactly. For this purpose assume that a change ψ in the adjoint vector results because of the error ξ, i.e.,

$$p = P + \psi \tag{12}$$

Since p can be expressed as a function of x

$$p(x) = p(X + \xi)$$

by expanding about the state X, and retaining only the first two terms, we obtain

$$p(x) = p(X) + \left[\frac{\partial p_j}{\partial x_l}\right]_{x=X} \xi$$

By (12) the first term $p(X)$ is the adjoint vector P of the simplified problem ; the second term is the vector ξ premultiplied by a gain matrix

$$M(X) = \left[\frac{\partial p_j}{\partial x_l}\right]_{x=X}$$

Thus (12) can be written

$$p(x) = P(X) = M(X)\,\xi \tag{13}$$

and thus

$$\psi(t) = M(t)\,\xi(t) \tag{14}$$

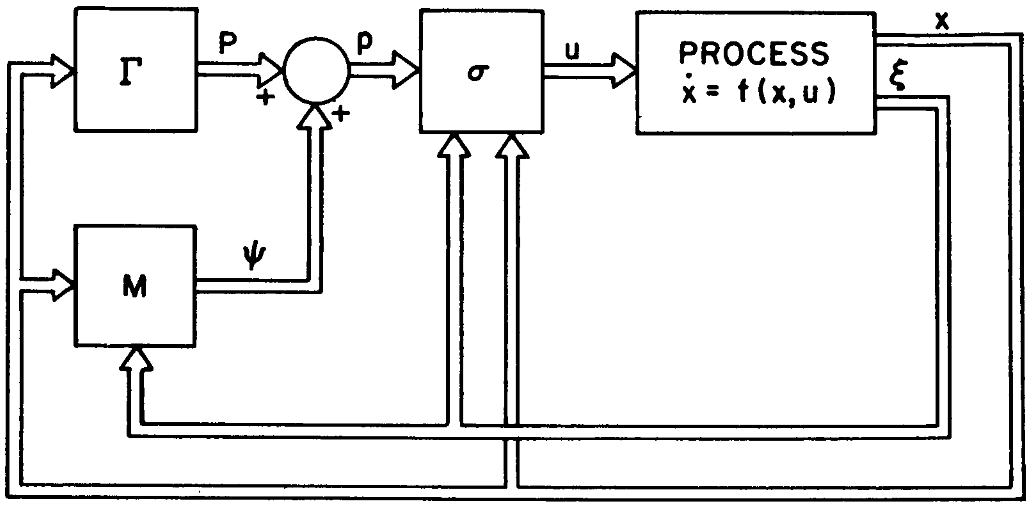
The structure of a control system based on the above approximation is shown in Figure 2. The suboptimum controller comprises three units: the σ - unit which is the same as determined for Figure 1 by maximizing h, the unit Γ which transforms X into P, and the gain unit $M(X)$ by which ξ is multiplied to yield a correction to P.

For this method to be useful, one must be able to compute the matrix $M(X)$ without prior knowledge of $p = \gamma(x)$, since if γ were known there would be not need to use the approximate control law (13). To obtain a relation for M, differentiate (13) with respect to time:

$$\dot{p} = \dot{P} + M\,\dot{\xi} + \dot{M}\,\xi \tag{15}$$

Likewise

$$\dot{x} = \dot{X} + \dot{\xi}$$



STRUCTURE OF
QUASI-OPTIMUM CONTROL SYSTEM
FIGURE 2

•Substituting these relations into the canonical equations (3) and (4) and expanding about the state and the adjoint for the simplified process gives

$$\dot{X} + \dot{\xi} = h_p = h_P + (H_{XP} + H_{PP}M)\,\xi + O(\xi^2)$$
$$\dot{P} + \dot{M}\,\xi + M\dot{\xi} = -h_x = -h_X - (H_{XX} + H_{PX}M)\xi + O(\xi^2) \tag{16}$$

where,

$$H_{XP} = \left[\frac{\partial^2 h}{\partial x_j\,\partial p_l}\right]_{x=X} \qquad H_{PX} = \left[\frac{\partial^2 h}{\partial p_j\,\partial x_l}\right]_{x=X} = H'_{XP}$$

$$H_{PP} = \left[\frac{\partial^2 h}{\partial p_j\,\partial p_l}\right]_{x=X} \qquad H_{XX} = \left[\frac{\partial^2 h}{\partial x_j\,\partial x_l}\right]_{x=X}$$

Upon use of (11), and after dropping terms of $O(\xi^2)$, (16) reduces to

$$\dot{\xi} = (H_{XP} + H_{PP}M)\,\xi \tag{17}$$

$$M\dot{\xi} + \dot{M}\xi = -(H_{XX} = H_{PX}M)\,\xi \tag{18}$$

Substitution of (17) into (18) gives:

$$(M + MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX})\,\xi = 0$$

If this relationship is to hold for any ξ , the matrix M must satisfy the matrix Riccati equation:

$$-\dot{M} = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} \tag{19}$$

It is evident that if M is a solution to (19) then M' is a solution to (19); thus, the solution to (19) can be a symmetric matrix.

1.3 SOLUTION OF THE RICCATI EQUATION

To use the technique described above, it is necessary to obtain the matrix M as a function of the state X of the simplified dynamic system. Solution of the Riccati equation (19) will give M as a function of the initial state of the simplified process $X(t)$ (since the coefficients of the partial derivative matrices on the right-hand side of (19) depend on $X(t)$) and time t. In an autonomous system, p can be expressed as a function of x only. Consequently the partial derivatives of p are not explicit functions of time and the dependence of M on time must be eliminated. This can be accomplished by expressing t as a function of X using the solution of the simplified system.

$$\dot{X} = F(X, \ u(P, \ X))$$

The general problem of solving the Riccati equation (19) remains. Three methods of determining the matrix M which were studied are described below.

(a) General Solution - A general solution to (19) can be obtained as follows. By substituting (19) into (16) we obtain the system

$$\begin{aligned} \dot{\xi} &= H_{XP}\xi + H_{PP}\psi \\ \dot{\psi} &= -H_{XX}\xi - H_{PX}\psi \end{aligned} \tag{20}$$

when the higher-order terms are dropped. This is a linear system whose solution can be expressed as

$$\begin{aligned} \xi(T) &= \Phi_{11}(T, \ t) \ \xi(t) + \Phi_{12}(T, \ t) \ \psi(t) \\ \psi(T) &= \Phi_{21}(T, \ t) \ \xi(t) + \Phi_{22}(T, \ t) \ \psi(t) \end{aligned} \tag{21}$$

where

$$\Phi(T, \ t) = \begin{bmatrix} \Phi_{11}(T, \ t) & \Phi_{12}(T, \ t) \\ \Phi_{21}(T, \ t) & \Phi_{22}(T, \ t) \end{bmatrix} \tag{22}$$

is the "transition matrix" corresponding to:

$$\mathcal{A} = \begin{bmatrix} H_{XP} & H_{PP} \\ -H_{XX} & -H_{PX} \end{bmatrix}$$

Equations (21) are actually $2(n + 1)$ equations in $n + 1$ unknowns. To solve we need $(n + 1)$ relations in addition to (21). These relations come from the boundary conditions.

Suppose that for the exact problem the boundary conditions at $\tau = T$ are given by (2) and (7).

If in the simplified process the boundary conditions are satisfied at time T, then in the exact problem these conditions must be satisfied at $T + dT$.

By expanding the exact state and adjoint about the time T and dropping second-order infinitesimals, we obtain

$$\begin{aligned} x(T + dT) &= x(T) + \dot{x}(T)\, dT \\ &= X(T) + \xi(T) + \dot{X}(T)\, dT \end{aligned} \tag{23a}$$

$$\begin{aligned} p(T + dT) &= p(T) + \dot{p}(T)\, dT \\ &= P(T) + \psi(T) + \dot{P}(T)\, dT \end{aligned} \tag{23b}$$

Substituting (23a) in (2) and expanding about the state of the simplified process gives

$$\varphi(X(T)) + \Phi\xi(T) + \Phi\dot{X}(T)\, dT = 0 \tag{24a}$$

Similarly, for the adjoint we have

$$P(T) + \psi(T) + \dot{P}(T)\, dT = \begin{bmatrix} -1 \\ ---- \\ \Phi'\lambda \end{bmatrix} \tag{24b}$$

Since the simplified problem has been assumed to satisfy the boundary conditions of the same form, i.e., $\varphi(X(T)) = 0$ and $P(T) = \begin{bmatrix} -1 \\ ---- \\ \Phi'\Lambda \end{bmatrix}$, then (24a) and (24b) reduce to the $n + 1$ independent equations

$$\Phi\,[\xi(T) + \dot{X}(T)\, dT] = 0 \tag{25a}$$

$$\psi(T) + \dot{P}(T)\, dT = \Phi'\eta \tag{25b}$$

where

$$\eta = \lambda - \Lambda$$

Finally, we must have

$$\begin{aligned} dH &= \xi' \frac{\partial H}{\partial X} + \psi' \frac{\partial H}{\partial P} \\ &= -\dot{P}'\xi + \dot{X}'\psi = 0 \end{aligned} \tag{26}$$

Equations (25a), (25b) and (26) give a total of $n + 2$ relations. Since dT is an additional variable, there are just enough equations needed to solve (21) for $\varphi(t)$ as a function of $\xi(t)$ and thereby obtain $M(t)$.

In most cases, the linear differential equations (20) have time-varying coefficients and as a result cannot be solved analytically. Hence, it becomes necessary either to approximate the solution to the Riccati equation or the integrate (19) numerically.

(b) Numerical Solution - Numerical integration of the Riccati equation requires that boundary conditions (25a) and (25b) be translated into conditions on $M(T)$. Consequently, (19) must be integrated backwards in time starting at $\tau = T$. Part of the complexity of this problem arises because the matrix $M(\tau)$ may not exist at $\tau = T$, hence, the boundary conditions cannot be translated directly into conditions on $M(T)$. This problem may be circumvented by expressing $M(t)$ in the form

$$M(t) = S(t) - R(t)Q^{-1}(t)R'(t) \tag{27}$$

integrating systems of differential equations for S, Q and R for a small time Δ backwards from T and using the results to compute $M(T - \Delta)$.

Let the matrix Riccati equation be of the form

$$-\dot{M} = MA + A'M + MBM + C \tag{28}$$

where $$A = H_{XP}, \; B = H_{PP}, \; C = H_{XX}$$

Suppose we have any solution S to the matrix Riccati equation (28), i.e.,

$$-\dot{S} = SA + A'S + SBS + C \tag{29}$$

Then the desired matrix M can be expressed in terms of S by (27),

$$M = S - RQ^{-1}R'$$

where Q is a symmetric $(n + 1) \times (n \times 1)$ matrix and R is a rectangular $(n + 1) \times (s + 1)$ matrix. Matrices Q and R in turn satisfy the differential equations

$$-\dot{R} = (A' + SB)R \tag{30}$$

$$-\dot{Q} = R'BR \tag{31}$$

To verify (27) differentiate both sides of (27) and substitute into (19). The result is

$$-\dot{S} + \dot{R}Q^{-1}R' - RQ^{-1}\,\dot{Q}Q^{-1}R' + RQ^{-1}\dot{R}'$$

$$= (S - RQ^{-1}R')A + A'(S - RQ^{-1}R') + (S - RQ^{-1}R')\,B(S - RQ^{-1}R') + C$$

$$= (SA + A'S) + C - RQ^{-1}R'(A + BS) - (A' + SB)\,RQ^{-1}R' + RQ^{-1}BRQ^{-1}R'$$

which, upon use of (30) and (31), is an identity. This result is a generalization of the result obtained by McReynolds and Bryson [9].

If $Q(T)$ is singular, then $M(T)$ cannot be expressed in the form of (27) moreover, $M(T)$ may not exist. However $M(T - \Delta)$ can be computed by integrating (29), (30) and (31) (backwards) from $\tau = T$ to $\tau = T - \Delta$ until $Q(T - \Delta)$ is nonsingular and then using (27), i.e.,

$$M(T - \Delta) = S(T - \Delta) - R(T - \Delta)\,Q^{-1}(T - \Delta)\,R'(T - \Delta) \tag{32}$$

From $\tau = T - \Delta$ back, (32) can be used as a starting condition for (19).

The boundary conditions on S, R and Q can be determined by taking the solution of the auxiliary system (20) to be of the form

$$\psi(\tau) = S(\tau)\,\xi(\tau) + R(\tau)\,\mu(\tau) \tag{33}$$

where
$$Q(\tau)\,\mu(\tau) = -R'(\tau)\,\xi(\tau) \tag{34}$$

If we define

$$\mu(T) = \begin{array}{cc} \left[\begin{array}{c} \eta \\ \hline dT \end{array}\right] & \begin{array}{c} s \\ 1 \end{array} \end{array}$$

$$R(T) = \begin{array}{cc} \left[\begin{array}{c|c} \bar{R}(T) & r(T) \end{array}\right] & n+1 \\ \begin{array}{cc} \quad s \quad & \quad 1 \quad \end{array} & \end{array} \tag{35}$$

$$Q(T) = \begin{array}{cc} \left[\begin{array}{c|c} \bar{Q}(T) & q(T) \\ \hline q'(T) & c(T) \end{array}\right] & \begin{array}{c} s \\ 1 \end{array} \end{array}$$

then at $\tau = T$, (33) and (34) are

$$\psi(T) = S(T)\,\xi(T) + \bar{R}(T)\,\eta + r(T)\,dT \tag{36}$$

$$\bar{Q}(T)\eta + q(T)\,dT = -\bar{R}(T)\,\xi(T) \tag{37}$$

$$q'(T)\eta + c(T)\,dT = -r'(T)\,\xi(T) \tag{38}$$

The unknowns $S(T)$, $\bar{R}(T)$, $r(T)$, $\bar{Q}(T)$, $q(T)$ and $c(T)$ must be chosen to satisfy the boundary conditions (25a), (25b) and (26).

It is seen that (25b) is satisfied by making

$$
\begin{aligned}
S(T) &= 0 \\
\bar{R}(T) &= \Phi' \\
r(T) &= -\dot{P}(T)
\end{aligned}
\tag{39}
$$

With this choice of $\bar{R}(T)$ and $r(T)$, (37) and (38),

$$\bar{Q}(T)\eta + q(T)\, dT = - \Phi\, \xi(T) \tag{40a}$$

$$q'(T)\eta + c(T)\, dT = \ddot{P}(T)\xi(T) \tag{40b}$$

Condition (25a) is satisfied by making

$$q(T) = \Phi \dot{X}(T)$$

$$\bar{Q}(T) = 0$$

then (40b) becomes

$$\dot{X}'(T)\Phi'\eta + c(T)\, dT = \ddot{P}(T)\xi(T)$$

which upon use of (25b) and (26) becomes

$$\dot{X}'\left[\psi(T) + P(T)\, dT\right] + c(T)\, dT = - \dot{X}'(T)\psi(T)$$

which is an identity for

$$c(T) = -\dot{X}'(T)\dot{P}(T)$$

In summary, the required $M(t)$ can be expressed as

$$M = S - RQ^{-1}R'$$

where S, Q, and R satisfy (29), (30) and (31), respectively, and where the following terminal conditions apply

$$
\begin{aligned}
S(T) &= 0 \\
R(T) &= \left[\, \Phi' \;\vdots\; -\dot{P}(T) \,\right]
\end{aligned}
\tag{41}
$$

$$Q(T) = \left[\begin{array}{c|c} 0 & \Phi X(T) \\ \hline \dot{X}'(T)\,\Phi' & -\dot{X}'(T)P(T) \end{array}\right]$$

(c) Approximation - In some cases, it may valid to approximate the solution to the Riccati equation. Since the use of (14) to generate the adjoint vector p is already an approximation, this "compounds the approximation," but still may give satisfactory results. The obvious approximate solution of (19) is the asymptotic solution obtained by setting the left-hand side to zero. The solution to the resulting equation

$$MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} = 0 \tag{42}$$

is the desired approximate value of M. (This is similar to the proposed solution of Pearson [1].) An illustration of this technique is given in Section 2.2. Unfortunately, this technique may not always work, because (42) may not possess a nontrivial solution.

1.4 INTERPRETATION OF M

When the adjoint vector p can be interpreted as the negative gradient of the optimum performance function [7]

$$p = -\text{grad}_x V \tag{43}$$

where,

$$V(x) = \min_{u \in \Omega} x_0(T)$$

obtained as the solution to the optimization problem, the Hamiltonian is given by:

$$-h = f'(x, u)\ \text{grad}_x V$$

and (5) becomes the Hamilton-Jacobi differential equation.

Upon differentiation of p given by (43) we obtain

$$\dot{p} = \left[\frac{\partial p_j}{\partial x_j}\right] x = -\frac{\partial^2 V}{\partial x^2}\dot{x} \tag{44}$$

where,

$$\frac{\partial^2 V}{\partial x^2} = \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]$$

Comparing (44) with (13) it is found that

$$M(X) = -\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=X}$$

the negative Hessian of V with respect to the state x. In this case $M(X)$, being a matrix of second partial derivatives of a scalar function, of course, must be symmetric.

Another relation which must be satisfied by $M(X)$ is obtained by setting $x = X$ in (44). In this case $p = P$ and we obtain:

$$\dot{P} = M(X)\dot{X} \tag{45}$$

This means that if P and X are the solutions to the simplified problem, the desired matrix $M(X)$ must satisfy the above equation. Thus (45) can be employed as a check on the calculation of $M(X)$.

.1.5 ALTERNATIVE TECHNIQUES

Alternative approaches to computing a quasi-optimal control may be developed by making use of the relations

$$p(T) = P(t) + \psi(t)$$

and

$$\psi(t) = M(t)\xi(t)$$

A technique based on this development consists of storing trajectories $X(t)$, $P(t)$ and $M(t)$ in the controller (or by generating these quantities by integrating (11) and (19) with nominal initial conditions $X(0)$, $P(0)$, $M(0)$) and computing the control

$$u(t) = \sigma(X(t), \; P(t) + M(t)\xi(t)) \qquad 0 < t < T$$

as shown in Figure 3.

This technique is very similar to the second-variation techniques of Kelley [3] and of Breakwell, Bryson and Speyer [2] , except that here the adjoint vector $P(t)$ rather than the control $u(t)$ is stored. The difference is relatively insignificant when u is a continuous function of p, but is of major significance when u is a bounded, discontinuous function of p. For example, if the optimum control law is of the form $u = \text{sgn}\,(c'p)$ then there is no reasonable way to make a linear correction to the nominal control u, but $u = \text{sgn}\,[c'(P + M\xi)]$ is an entirely reasonable control law.

A second method of employing the suboptimum technique is based on (45).

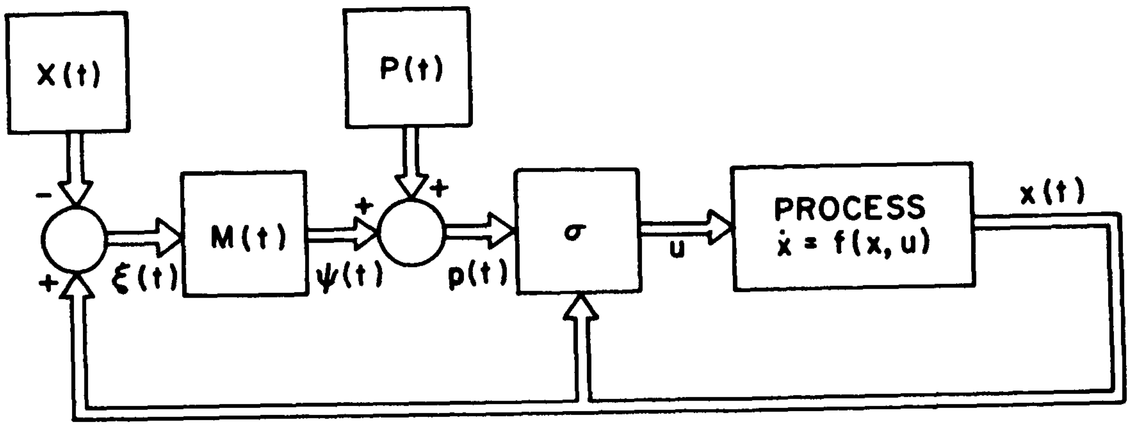
The adjoint vector can be obtained by integration of (45);

$$p = p(0) + \int_0^t M\dot{x}\, d\tau \tag{47}$$

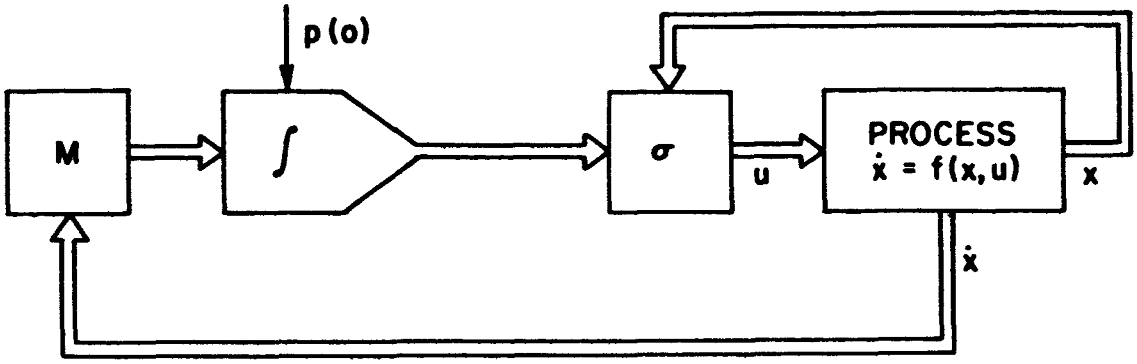
This relation leads to a control system with the configuration shown in Figure 4(a). It is noted that the derivatives of the state variables instead of the state variables themselves are the quantities fed back. Hence this technique is particularly applicable to problems in inertial guidance, where the principal sensors are accelerometers.

In the event that $\dot{x}$ cannot be sensed, an alternative configuration can be obtained by partial integration of (47):

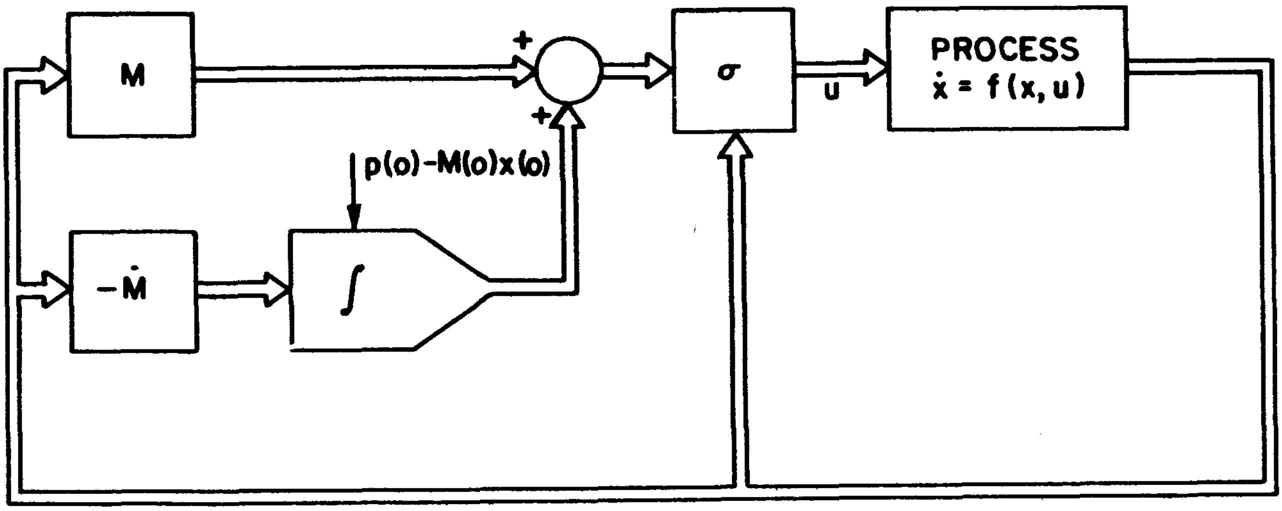
$$p = p(0) + Mx - M(0)x(0) - \int_0^t \dot{M}x\, d\tau \tag{48}$$



SUBOPTIMUM CONTROL BASED ON
STORED NOMINAL TRAJECTORY
FIGURE 3



(a) $\dot{x}$ MEASURABLE



(b) $\dot{x}$ NOT MEASURABLE

SUBOPTIMUM CONTROL BASED ON p(o), M(o)
FIGURE 4

The right-hand side of (19) is used for $-M$ in (48). The control system configuration corresponding to (48) is shown in Figure 4(b); it is seen that only the state x is required in the controller.

In either implementation the matrix M would be generated by real-time integration of (19) with the nominal initial condition $M(0)$, and the nominal initial adjoint state $P(0)$ would be used. Thus, to achieve near-optimum performance, the actual initial state $x(0)$ should be reasonably close to the nominal initial state $X(0)$ for which $M(0)$ and $P(0)$ were computed. If the closed-loop system is asymptotically stable, however, the effects of using initially incorrect values of $M(0)$ and $p(0)$ will be only transient.

An example of the application of the control technique of Figure 3, in which the control variable is bounded, is discussed in "A Unified Approach to Suboptimum Control" by B. Friedland and P.E. Sarachik, to be presented at the Third Congress, International Federation of Automatic Control, London, England June 20-25, 1966.

1.6 BOUNDED CONTROL VARIABLES

When there are "hard" constraints present, such as $u_j \le K_j$, there is a further difficulty in computing the matrix M. The difficulty arises because H_p generally turns out to be a discontinuous function of the adjoint variables -- typically $u = \text{sgn}\,(c'p)$ -- and thus the partial derivatives do not strictly exist at the points of discontinuity, i.e., upon the switching surface. As Kalman [7] and others have pointed out, this behavior precludes identification of p with the negative of the gradient of V and also raises doubts about the validity of the Riccati equation (19) and the associated linear two-point boundary-value problem (20), (25) and (26).

As a consequence of the discontinuities in the control, some of the second partial derivatives of the Hamiltonian may not exist in certain regions of the state space. By applying the quasi-optimal procedure to the Bushaw problem, however, it was found that inclusion of the delta functions which arise from formally differentiating the discontinuous Hamiltonian function gives the correct answer. Details of this calculation are given in Section 3.1.

The use of delta functions can be probably be justified by a careful limiting procedure, i.e., by defining a sequence of problems, each with a constraint on the control variable which approaches closer to the bounded constraint than the previous, and then showing that there is a limiting solution which is the one obtained by use of delta functions.

1.7 QUASI-OPTIMUM CONTROL OF MILDLY NONLINEAR PROCESSES

Consider a special case of the general process

$$\dot{x} = Ax + \mu f(x) + Bu \tag{49}$$

with a performance criterion

$$V = \frac{1}{2} \int_t^T (x'Rx + u'Qu)\, d\tau \tag{50}$$

which is to be minimized, where T is fixed; Q is a positive-definite matrix; μ is a small parameter, and $f(x)$ is a nonlinear function which is twice differentiable with respect to all its arguments. When $\mu = 0$, (49) becomes a linear process and, for the quadratic performance criterion (50), a linear feedback law results. The problem we wish to treat here is the determination of an approximate control law when μ is small but not zero.

Exact Problem - The augmented state of the exact process is defined by

$$x = \{x_0, \tau, x, \mu\} \tag{51}$$

with the following auxiliary differential equations

$$\dot{x}_0 = \frac{1}{2}(x'Rx + u'Qu), \quad \dot{\tau} = 1, \quad \dot{\mu} = 0 \tag{52}$$

where x_0 is the performance, and x is the n-dimensional state vector.

The adjoint vector for this problem is defined by

$$\tilde{p} = \{p_0, p_\tau, p, p_\mu\} \tag{53}$$

and the Hamiltonian is

$$h = \frac{p_0}{2}(x'Rx + u'Qu) + p_\tau + p'(Ax + \mu f(x) + Bu) \tag{54}$$

Application of the maximum principle gives the optimum control law as

$$u = -\frac{1}{p_0} Q^{-1} B' p \tag{55}$$

and this control law results in the following expression for the Hamiltonian

$$h = \frac{p_0}{2} x'Rx - \frac{1}{2p_0} p'BQ^{-1}B'p + p_\tau + p'Ax + p'f(x)\mu$$

The canonical equations are thus

$$\dot{x} = h_p = \begin{bmatrix} \frac{1}{2}x'Rx + \frac{1}{2p_0^2}p'BQ^{-1}B'p \\ 1 \\ Ax + f(x)\mu - \frac{BQ^{-1}B'p}{p_0} \\ 0 \end{bmatrix} \qquad -\dot{p} = h_x = \begin{bmatrix} 0 \\ 0 \\ p_0Rx + [A' + \mu(\frac{\partial f}{\partial x})]\,p \\ p'f(x) \end{bmatrix} \tag{56}$$

The following boundary conditions apply at the terminal time T

$$\begin{array}{ll} x_0(T) = \text{minimum}; & p_0(T) = -1 \\ \tau(T) = T & p_t(T) \quad \text{free} \\ x(T) \quad \text{free} & p(T) = 0 \\ \mu \quad \text{free} & p_\mu(T) = 0 \end{array} \tag{57}$$

Simplified Problem - For the simplified problem we take $\mu = 0$, whence

$$X = \{X_0, \tau, X, 0\} \quad , \quad P = \{P_0, P_\tau, P, 0\}$$

The canonical equations (56) and (57) become

$$\dot{X}_0 = \frac{1}{2}(X'RX + \frac{1}{P_0^2} P'BQ^{-1}B'P) \quad , \quad \dot{\tau} = 1$$

$$\dot{X} = AX - BQ^{-1}B'P/P_0$$

$$\dot{P}_0 = 0 \quad , \quad P_\tau = 0$$

$$\dot{P} = -P_0RX - A'P$$

Since $\dot{P}_0 = 0$, $P_0 = \text{constant} = -1$ hence the equations for P and X are

$$\begin{aligned} \dot{X} &= AX + BQ^{-1}B'P \\ \dot{P} &= RX - A'P \end{aligned} \tag{59}$$

The solution to (59) is

$$X(\tau) = \Phi_{11}(\tau, t)X(t) + \Phi_{12}(\tau, t)P(t) \tag{60}$$

$$P(\tau) = \Phi_{21}(\tau, t)X(t) + \Phi_{22}(\tau, t)P(t) \tag{61}$$

where

$$\Phi(\tau, t) = \begin{bmatrix} \Phi_{11}(\tau, t) & \Phi_{12}(\tau, t) \\ \Phi_{21}(\tau, t) & \Phi_{22}(\tau, t) \end{bmatrix}$$

is the fundamental matrix for the linear time-invariant system (59).

Applying the boundary condition $P(T) = 0$, (61) gives rise to

$$P(t) = -\Phi_{22}^{-1}(T, t)\Phi_{21}(T, t)X(t)$$

or, more generally,

$$P(\tau) = K(T, \tau)X(\tau) \tag{62}$$

where

$$K(T, \tau) = -\Phi_{22}^{-1}(T, \tau)\Phi_{21}(T, \tau) \tag{63}$$

The gain matrix $K(T, \tau)$ can also be obtained as the solution to a matrix Riccati equation: substitution of (62) in to (59) gives

$$\dot{X} = (A + BQ^{-1}B'K)X$$

$$\dot{P} = \dot{K}X + K\dot{X} = (R - A'K)X$$

for any X. Multiplying the first equation by K and subtracting results in the matrix Riccati equation

$$-\dot{K} = KA + A'K + KBQ^{-1}B'K - R \tag{64}$$

Since $\Phi_{21}(T, T) = 0$, the condition which must be satisfied by $K(T, \tau)$ is

$$K(T,T) = 0 \tag{65}$$

Quasi-Optimum Control - In accordance with the general theory, the approximate value of p is given by

$$\tilde{p} = \tilde{P} + \tilde{M}\xi$$

where $\tilde{M}$ is the correction matrix, and is conveniently partitioned as follows

$$\tilde{M} = \begin{bmatrix} m_{00} & m_{\tau 0} & \underline{m}'_{0x} & m_{\mu 0} \\ m_{\tau 0} & m_{\tau\tau} & \underline{m}'_{\tau x} & m_{\mu t} \\ \underline{m}_{0x} & \underline{m}_{tx} & M_{xx} & \underline{m}_{\mu x} \\ m_{\mu 0} & m_{\mu\tau} & \underline{m}'_{x\mu} & m_{\mu\mu} \end{bmatrix} \tag{66}$$

M_{xx} is an n x n matrix, $\underline{m}_{0x}$, $\underline{m}_{\mu x}$, and $\underline{m}_{\tau x}$ are n-vectors, and all the other quantities are scalars. The submatrices appearing in (66)--not all of which are required for suboptimum control law--are to be found with the aid of the auxiliary equations for ψ and ξ or by use of the matrix Riccati equation. The following matrices are required:

$$H_{XP} = \begin{bmatrix} 0 & 0 & X'R & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & f(X) \\ 0 & 0 & 0 & 0 \end{bmatrix} = H'_{PX}$$

$$H_{PP} = \begin{bmatrix} P'BQ^{-1}B'P & 0 & P'BQ^{-1}B' & 0 \\ 0 & 0 & 0 & 0 \\ BQ^{-1}B'P & 0 & BQ^{-1}B' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{67}$$

$$H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P_0R & (\partial f/\partial x)'P \\ 0 & 0 & P'(\partial f/\partial x) & 0 \end{bmatrix}$$

with $P_0 = -1$, $P = KX$

The auxiliary equations for $\tilde{\xi}$ and $\tilde{\psi}$ are

$$\begin{aligned} \dot{\tilde{\xi}} &= H_{XP}\tilde{\xi} + H_{PP}\tilde{\psi} \\ \dot{\tilde{\psi}} &= H_{XX}\tilde{\xi} + H_{PX}\tilde{\psi} \end{aligned} \tag{68}$$

It is convenient to partition $\tilde{\xi}$ and $\tilde{\psi}$ the same way as $\tilde{x}$ and $\tilde{p}$, i.e.

$$\begin{aligned} \tilde{\xi} &= [\xi_0, \xi_\tau, \xi, \xi_\mu] \\ \tilde{\psi} &= [\psi_0, \psi_\tau, \psi, \psi_\mu] \end{aligned} \tag{69}$$

In terms of the subvectors of ξ and ψ, the expressions of (20) become:

$$\dot{\xi}_0 = X'R\xi + P'BQ^{-1}B'P\psi_0 + P'BQ^{-1}B'\psi \tag{69a}$$

$$\dot{\xi}_\tau = 0 \tag{69b}$$

$$\dot{\xi} = A\xi + f(X)\xi_\mu + BQ^{-1}B'P\psi_0 + BQ^{-1}B'\psi \tag{69c}$$

$$\dot{\xi}_\mu = 0 \tag{69d}$$

$$\dot{\psi}_0 = 0 \tag{69e}$$

$$\dot{\psi}_\tau = 0 \tag{69f}$$

$$\dot{\psi} = R\xi - (\partial f/\partial x)'P\xi_\mu - RX'\psi_0 - A'\psi \tag{69g}$$

$$\dot{\psi}_\mu = -P'(\partial f/\partial x)\xi - f'(X)\psi \tag{69h}$$

The relevant boundary conditions are

$$\psi_0(T) = P_0(T)\ dT = 0 \tag{70a}$$

$$\xi_t(T) = \dot{\tau} dT = dT \tag{70b}$$

$$\psi(T) = \dot{P}(T)\, dT = RX(T)\, dT \tag{70c}$$

$$\psi_\mu(T) = \dot{P}_\mu(T)\, dT = 0 \tag{70d}$$

and $\dot{\tilde{P}}' \xi = \dot{\tilde{X}}' \psi$ or

$$X'(T)R\xi(T) = \frac{1}{2} X'(T)RX(T)\psi_0(T) + \psi_\tau + X'(T)A'\psi(T) \tag{70e}$$

From (69e) and (70a)

$$\psi_0(t) = 0$$

Consequently, the first row and column of the transition matrix and of the matrix M in (64) are zero. Also, from (69d)

$$\xi_\mu(\tau) = \text{const} = \mu$$

Hence, (69c) and (69g) can be written

$$\begin{aligned} \dot{\xi} &= A\xi + BQ^{-1}B'\psi + f(X)\mu \\ \dot{\psi} &= R\xi - A'\psi - (\partial f/\partial x)' P\mu \end{aligned} \tag{71}$$

It is observed that the matrix coefficients in the homogeneous form of (71) are the same as for the exact problem (59). The solution to (71) can be expressed in terms of the blocks of the fundamental matrix defined in (59) and (60):

$$\begin{aligned} \xi(\tau) &= \Phi_{11}(\tau, t)\xi(t) + \Phi_{12}(\tau, t)\psi(t) + \int_t^\tau \mu [\Phi_{11}(\tau, \lambda) f(X) - \Phi_{12}(\tau, \lambda)(\partial f/\partial x)' P]\, d\lambda \\ \psi(\tau) &= \Phi_{21}(\tau, t)\xi(t) + \Phi_{22}(\tau, t)\psi(t) + \int_t^\tau \mu [\Phi_{21}(\tau, \lambda) f(X) - \Phi_{22}(\tau, \lambda)(\partial f/\partial x)' P]\, d\lambda \end{aligned} \tag{72a}$$

Evaluating the second equation of (72a) at $\tau = T$ and employing (70b) and (70c), we obtain

$$\psi(t) = \Phi_{22}^{-1}(T, t)\Big[RX(T)\xi_t - \Phi_{21}(T, t)\xi(t) - \mu\int_t^T [\Phi_{21}(T, \lambda) f(X) - \Phi_{22}(T, \lambda)(\partial f/\partial x)' P]\, d\lambda\Big] \tag{72b}$$

Since, in accordance with (66),

$$\psi(t) = \underline{m}_{\tau x}\xi_{\tau}(t) + M_{xx}\xi(t) + \underline{m}_{\mu x}\mu \tag{73}$$

it follows from (72b) that

$$\underline{m}_{\tau x}(T, t) = \Phi_{22}^{-1}(T, t)RX(T) \tag{74}$$

$$M_{xx}(T, t) = -\Phi_{22}^{-1}(T, t)\Phi_{21}(T, t) = K(T, t)$$

$$\underline{m}_{\mu x}(T, t) = -\Phi_{22}^{-1}(T, t)\int_{t}^{T}[\Phi_{21}(T, \lambda)f(X) - \Phi_{22}(T, \lambda)(\partial f/\partial x)'P]\, d\lambda \tag{75}$$

Although the other components of $\tilde{M}$ defined in (66) can be obtained by use of (70d) and (70e), they are not needed for the quasi-optimum control law: only p_0 and p are needed. These are given by

$$p_0 = P_0 + (m_{00}, m_{\tau 0}, \underline{m}_{0x}, m_{\mu 0})\tilde{\xi}$$

$$p = P + (\underline{m}_{0x}, \underline{m}_{\tau x}, M_{xx}, \underline{m}_{\mu x})\tilde{\xi}$$

Since the first row and column of $\tilde{M}$ are zero

$$p_0 = P_0 = -1$$

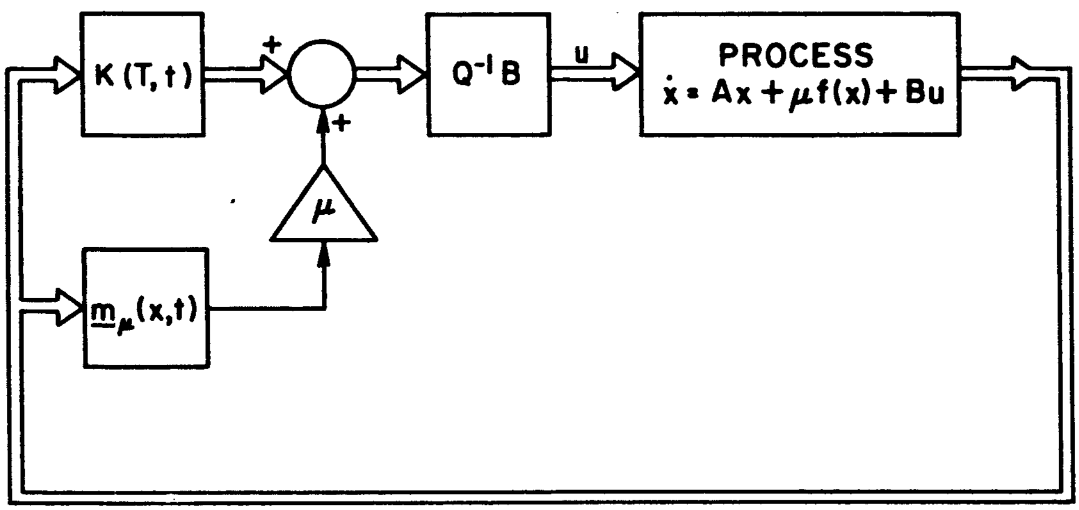
Also, since $P(t) = K(T, t)X(t)$ and $M_{xx} = K(T, t)$,

$$\begin{aligned} p &= K(T, t)[X + \xi] + \underline{m}_{\tau x}dT + \underline{m}_{\mu x}\mu \\ &= K(T, t)x + \underline{m}_{\tau x}dt + \underline{m}_{\mu x}\mu \end{aligned} \tag{76}$$

The second term on the right-hand side of (76) is a correction due to a change in time (clock error) and can be ignored. Consequently, from (53) the quasi-optimum control law is

$$u = Q^{-1}B'(K(T, \tau)x + \underline{m}_{\mu x}(x, \tau)\mu) \tag{77}$$

where $\underline{m}_{\mu x}$ is the nonlinear, vector-valued function of x defined by (75). The control system has the configuration of Figure 5.



QUASI-OPTIMUM CONTROL SYSTEM
FIGURE 5

Note that the nonlinear feedback is proportional to the parameter μ, so that the system is predominantly linear, with a small nonlinear correction.

Differential Equation for $\underline{m}_{\mu x}$ - Instead of having to evaluate $\underline{m}_{\mu x}$ by integration of (75), which requires the Φ_{21} and Φ_{22} matrices, it may be more convenient to find $\underline{m}_{\mu x}$ by integrating a differential equation. Provided $f(X)$ and $\partial f/\partial X$ have no impulse at $\tau = T$, (75) gives

$$\underline{m}_{\mu x}(T, T) = 0 \tag{78}$$

Using (73) and (69b),

$$\dot{\psi} = \dot{\underline{m}}_{\tau x}\xi_\tau + \dot{M}_{xx}\xi + M_{xx}\dot{\xi} + \dot{\underline{m}}_{\mu x}\mu$$

Substitution of this relation and ψ given by (73) into (71) results in

$$\dot{\xi} = A + BQ^{-1}B'(\underline{m}_{\tau x}\xi_\tau + M_{xx}\xi + \underline{m}_{\mu x}\mu) + f(X)\mu$$

$$\dot{m}_{\tau x}\xi_\tau + \dot{M}_{xx}\xi + M_{xx}\dot{\xi} + \dot{\underline{m}}_{\mu x}\mu = R\xi - A'(\underline{m}_{\tau x}\xi_\tau + M_{xx}\xi + \underline{m}_{\mu x}\mu) - (\partial f/\partial x)' P\mu$$

Multiplying the first by M_{xx}, subtracting the second, and using the fact that ξ_τ, ξ, and μ are arbitrary, leads to the following equations

$$-\dot{M}_{xx} = M_{xx}A + AM_{xx} + M_{xx}BQ^{-1}B'M_{xx} - R \tag{79}$$

$$-\dot{\underline{m}}_{\tau x} = (A' + M_{xx}BQ^{-1}B')\underline{m}_{\tau x} \tag{80}$$

$$-\dot{\underline{m}}_{\mu x} = (A' + M_{xx}BQ^{-1}B')\underline{m}_{\mu x} + M_{xx}f(x) + (\partial f/\partial x)P \tag{81}$$

Thus, since $M_{xx} = K$, (80) and (81) are differential equations by which $\underline{m}_{\tau x}$ and $\underline{m}_{\mu x}$ can be computed. It is of interest to note that the homogeneous form of (80) and (81) is

$$\dot{b} = -(A + BQ^{-1}B'K)'b \qquad\qquad b' = (\underline{m}_{\tau x}, \underline{m}_{\mu x})$$

which is the adjoint differential equation for the differential equation for the closed-loop simplified process.

• Application to a Linear Process – A special case of interest is when $f(x) = Fx$, that is, the nonlinear process (49) reduces to a linear process. In this case $\underline{m}_{\mu x}$ is a linear function of x, and the quasi-optimum control law (77) becomes

$$u = Q^{-1}B'(Kx + Lx\mu) = Q^{-1}B'(K + \mu L)x \tag{82}$$

where

$$\underline{m}_{\mu x} = Lx$$

Thus the quasi-optimum control law is linear in x. This is not surprising, since the exact optimum control law is also linear in x. In fact the exact control law is given by

$$u = Q^{-1}B'Sx \tag{83}$$

where S satisfies the matrix Riccati equation (64), with $S(T, T) = 0$ and A in (64) replaced by $A + \mu F$. Thus

$$-\dot{S} = S(A + \mu F) + (A' + \mu F')S + SBQ^{-1}B'S - R \tag{84}$$

Comparing (82) and (83) however, we must have, to first order in μ,

$$S = K + \mu L \tag{84}$$

or

$$-\dot{K} - \mu\dot{L} = (K + \mu L)(A + \mu F) + (A' + \mu F')(K + \mu L) + (K + \mu L)BQ^{-1}B'(K + \mu L) - R$$

On expanding the right-hand side of this equation and equating the constant terms and the coefficients of μ on both sides we obtain

$$\begin{aligned} -\dot{K} &= KA + A'K + KBQ^{-1}B'K - R \\ -\dot{L} &= L(A + BQ^{-1}B'K) + (A' + KBQ^{-1}B')L + KF + F'K \end{aligned} \tag{85}$$

Thus L is given by a linear equation of the Riccati type, with $KF + F'K$ as the forcing function. Now consider (81) with $M_{xx} = K$, $\underline{m}_{\mu x} = Lx$, $P = Kx$ and $f(x) = Fx$. We obtain

$$-L\dot{x} - \dot{L}x = (A' + KBQ^{-1}B')Lx + (KF + F'K)x \tag{86}$$

but

$$\dot{x} = (A + BQ^{-1}B'K)x \tag{87}$$

Hence (86) and (87) are equivalent to (85). Thus the general expressions for $\underline{m}_{\mu x}$ given by (75) and (81) can be viewed as an extension of the linear correction considered here.

Although the exact problem when $f(x) = Fx$ can be solved by use of (84), there are possible advantages to be gained by using $S = K + \mu L$. Consider a flexible booster, for example. The process is linear (for small angles) but of quite high order. If the stiffness is high, however, then a rigid body model of low order might be a reasonable first approximation, μ can be regarded as the reciprocal of stiffness, and L then becomes the correction due to flexibility. This technique thus permits the separation of the "rigid body gains" from the "flexible mode" gains.

Asymptotic Solution - If the terminal time T is infinite, then the gain matrix K for the simplified problem becomes a constant K_∞ (when the process is observable and controllable) that is the asymptotic solution to the Riccati equation (64) can be obtained, and by setting the left-hand side of (64) sequal to zero. The asymptotic expression for $\underline{m}_{\mu x}$, if it exists, is given by the limit as $T \to \infty$ of (75). It is tempting to assume that the asymptotic solution for $\underline{m}_{\mu x}$ can be obtained by setting M_{xx} equal to K_∞ and $\underline{m}_{\mu x} = 0$ in (81), and solving for $\underline{m}_{\mu x}$. Unfortunately this procedure is incorrect, since in the case $\underline{m}_{\mu x} = Lx$, it results in a L matrix which is not symmetric. The reason is that (81), with $M_{xx} = K_\infty$, is the adjoint to a stable system (the closed loop system) which does not possess an asymptotic solution. To find the asymptotic solution, it is necessary to solve (79) and (81) concurrently. It is noted that $\underline{m}_{\mu x}(\infty, \tau)$ depends on $x(\tau)$, so it is not feasible to solve (79) and (81). The integral form (75) however, may be capable of being evaluated if $f(x)$ is a simple analytic function.

Part 2

APPLICATIONS

In order to verify the validity of the quasi-optimum control technique and to obtain some qualitative insight into some of the difficulties and limitations of the method, a number of "practical" problems to which the technique appears to be applicable were studied. The word "practical" is enclosed in quotation marks here to emphasize that even the equations (1) for the exact model entailed a considerable simplification of the actual physical behavior of the process; the simplified model (10) is a still further simplification. In all cases considered the further simplification led to a lower-order system of differential equations.

No theoretical difficulties were encountered in any of the examples studied; the algebraic calculations, however, although straightforward, were quite tedious and involved. Consequently progress was slow and calculations had to be checked frequently.

The following problems were considered:

1. Bushaw's Problem
2. Minimum-Time, Bounded Acceleration Rendezvous in Free Space
3. Flexible Booster Attitude Control
4. Time-Optimum 3-Axis Attitude Control of a Space Vehicle
5. Minimum-Miss Distance Maneuvering Reentry
6. Adaptive Control

2.1 BUSHAW'S PROBLEM

The first illustrative example is the "classical" minimum time problem for a plant having the transfer function $1/s(s+a)$ with a bounded control variable.

Exact Problem - The sytem is governed by the following differential equations

$$\begin{aligned} \dot{x}_0 &= 1 \\ \dot{x}_1 &= -x_1 x_3 + u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= 0 \end{aligned} \tag{1-1}$$

It is desired to minimize $x_0(T)$ with

$$x_1(T) = x_2(T) = 0 \text{ and } |u| \leq 1 \tag{1-2}$$

The Hamiltonian for the complete problem is:

$$h = p_0 + p_1(u - x_1 x_3) + p_2 x_1 \tag{1-3}$$

and the maximum principle gives

$$u = \operatorname{sgn} p_1(t) \tag{1-4}$$

The corresponding adjoint equations are

$$\begin{aligned} \dot{p}_0 &= 0 \\ \dot{p}_1 &= p_1 x_3 - p_2 \\ \dot{p}_2 &= 0 \\ \dot{p}_3 &= p_1 x_1 \end{aligned} \tag{1-5}$$

Simplified Problem - Since $\dot{x}_3 = 0$, $x_3 = a =$ constant from the exact problem. The simplified problem is obtained by taking $x_3 = a = 0$, to which the transfer function $1/s^2$ corresponds. Thus the equations for the simplified problem are (1-1)-(1-5) with $x_3 = 0$. Hence

$$p_0 = P_0 = -1 = \text{constant}$$

$$p_1 = P_1 = P_{10} - P_2 t\,,\ P_{10} = \text{constant} \tag{1-6}$$

$$p_2 = P_2 = \text{constant}$$

where t is time measured from the (arbitrary) starting instant. Thus,

$$u = \operatorname{sgn}(P_{10} - P_2 t)$$

$$= \begin{cases} U, & t < P_{10}/P_2 = t_s \\ -U, & t > P_{10}/P_2 \end{cases} \tag{1-7}$$

where,

$$U = \operatorname{sgn}(P_{10}/P_2) = \pm\ 1$$

Substituting (1-7) into (1-1) and integrating to the terminal time T, results in the following relations.

$$X_1(T) = X_{10} + U(2t_s - T) = 0$$

$$X_2(T) = X_{20} + X_{10}T + U\,\frac{T^2}{2} - U(T - t_s)^2 = 0 \tag{1-8}$$

The simultaneous solution of (1-8) for the time-to-go T and t_s gives, after dropping the subscript 0,

$$T = -UX_1 + 2\left(\frac{1}{2}\,X_1^2 - UX_2\right)^{\frac{1}{2}}$$

$$t_s = \frac{P_1}{P_2} = -UX_1 + \left(\frac{1}{2}\,X_1^2 - UX_2\right)^{\frac{1}{2}} \tag{1-9}$$

But, from (1-3),

$$H = -1 + P_1 U + P_2 X_1 = 0 \tag{1-10}$$

which, together with (1-9) yields

$$P_1 = U - \frac{X_1}{\left(\frac{1}{2}X_1^2 - UX_2\right)^{\frac{1}{2}}}$$

$$P_2 = \frac{U}{\left(\frac{1}{2}\,X_1^2 - UX_2\right)^{\frac{1}{2}}} \tag{1-11}$$

The switching curve is given by $P_1 = 0$, or, from (1-11)

$$X_2 = -\frac{1}{2} U X_1^2 = \pm \frac{1}{2} X_1^2 \qquad (1\text{-}12)$$

as is well-known.

Quasi-Optimum Control Law - The quasi-optimum control law will be taken as

$$u = \operatorname{sgn}(P_1 + m_{13} x_3) \qquad (1\text{-}13)$$

where P_1 is the component of the adjoint vector corresponding to x_1 in the simplified problem. Next, we wish to compute the gain matrix M. The matrices appearing in the Riccati equation (19) are obtained by performing the required partial differentiations on the exact Hamiltonian (1-3) and evaluating at $x = X$. The results are:

$$H_{XP} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H_{PX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -X_1 & 0 & 0 \end{bmatrix}$$

$$H_{PP} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\delta(P_1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_1 \\ 0 & 0 & 0 & 0 \\ 0 & -P_1 & 0 & 0 \end{bmatrix}$$

where $P_1 = P_1(t)$ is given by (1-6), and

$$X_1(t) = \begin{cases} X_{10} + Ut & t \le t_s \\ X_1(t_s) - Ut & t \ge t_s \end{cases} \qquad (1\text{-}14)$$

The term $2\delta(P_1)$ is an impulse (delta function) occurring at the instant that $P_1(t) = 0$, i.e., at $t = t_s$. It is the treatment of this impulsive term that we wish to illustrate in particular by this example. The gain matrix $M(t)$ will be computed by utilizing the auxiliary system (20) for the Riccati equation along with the transition matrix (22). There is no loss of generality in taking the initial time $t = 0$. Hence the fundamental matrix can be written

$$\Phi(T, 0) = \Phi(T, t_s^+)\Phi(t_s^+, t_s^-)\Phi(t_s^-, 0) \quad 0 < t_s < T \qquad (1\text{-}15)$$

Were it not for the impulsive term, $\Phi(t_s^+, t_s^-)$ would be the identity matrix, but in the present case $\Phi(t_s^+, t_s^-)$ contains off-diagonal terms resulting from the impulse.

To compute $\Phi(T, t_s^+)$ and $\Phi(t_s^-, 0)$, we can ignore the impulsive term. To facilitate the computation of these matrices we write (20) in component form

$$
\begin{aligned}
\dot{\xi}_0 &= 0 & \dot{\psi}_0 &= 0 \\
\dot{\xi}_1 &= -X_1\xi_3 + 2\delta(P_1(t))\psi_1 & \dot{\psi}_1 &= P_1\xi_3 - \psi_2 \\
\dot{\xi}_2 &= \xi_1 & \dot{\psi}_2 &= 0 \\
\dot{\xi}_3 &= 0 & \dot{\psi}_3 &= P_1\xi_1 + X_1\psi_1
\end{aligned}
$$

These equations can be integrated with little difficulty from t to t_s^- and from t_s^+ to T. The solutions can be expressed as

$$\xi(t_s^-) = \Phi_{11}(t_s, 0)\xi(0) + \Phi_{12}(t_s, 0)\psi(0)$$

$$\psi(t_s^-) = \Phi_{21}(t_s, 0)\xi(0) + \Phi_{22}(t_s, 0)\psi(0)$$

where

$$\Phi_{11}(t_s, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -X_{10}t_s - Ut_s^2/2 \\ 0 & t_s & 1 & -X_{10}t_s^2 - Ut_s^3/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{12}(t_s, 0) = 0$$

$$\Phi_{21}(t_s, 0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2t_s^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & P_2t_s^2/2 & 0 & t_s^3/6 \end{bmatrix}$$

$$\Phi_{22}(t_s, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -t_s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & X_{10}t_s + Ut_s^2/2 & -X_{10}t_s^2/2 - Ut_s^3/6 & 1 \end{bmatrix}$$

Likewise

$$\xi(T) = \Phi_{11}(\tau)\,\xi(t_s^+) + \Phi_{12}(\tau)\,\psi(t_s^+)$$

$$\psi(T) = \Phi_{21}(\tau)\,\xi(t_s^+) + \Phi_{22}(\tau)\,\psi(t_s^+)$$

where $\tau = T - t_s$

and

$$\Phi_{11}(\tau) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -U\tau^2/2 \\ 0 & \tau & 1 & -U\tau^3/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{12}(\tau) = 0$$

$$\Phi_{21}(\tau) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_2\tau^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & -P_2\tau^2/2 & 0 & -UP_2\tau^4/6 \end{bmatrix}$$

$$\Phi_{22}(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\tau & 0 \\ 0 & 0 & 1 & 0 \\ 0 & U\tau^2/2 & -U\tau^3/6 & 1 \end{bmatrix}$$

Next we must calculate the transition matrix $\Phi(t_s, t_s^-)$.

The impulsive term appears in only one differential equation of the set (1-16):

$$\xi_1 = X_1\xi_3 + 2\delta(P_1(t))\psi_1$$

Integrating we obtain

$$\xi_1(t_s^+) = \xi_1(t_s^-) - \int_{t_s^-}^{t_s^+} X_1(t)\,\xi_3(t)\,dt + \int_{t_s^-}^{t_s^+} 2\delta(P_1(t))\,\psi_1(t)\,dt \qquad (1\text{-}17)$$

The integrand of the first integral on the right-hand side of (1-16) is continuous and hence the integral vanishes as $t_s^+ \to t_s^-$. To evaluate the second integral we employ the formula [8]

$$\delta(P_1(t)) = \frac{\delta(t - t_s)}{|\dot{P}_1(t_s)|} \qquad t_s^- < t < t_s^+$$

But $|\dot{P}_1(t_s)| = UP_2 = P_2/U$ from (1-11). Hence (1-17) becomes

$$\xi_1(t_s^+) = \xi_1(t_s^-) + \frac{2U}{P_2}\psi_1(t_s^-)$$

(since $\psi_1(t_s^-) = \psi_1(t_s^+)$). Thus

$$\Phi_{11}(t_s^+, t_s^-) = \Phi_{22}(t_s, t_s^-) = 1$$

$$\Phi_{21}(t_s^+, t_s^-) = 0$$

and

$$\Phi_{12}(t_s^+, t_s^-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2U/P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, on performing the matrix multiplication indicated by (1-15) we obtain $\Phi(T, 0)$ with components given by

$$\Phi_{11}(T, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -X_{10}T + UX_{10}/2 \\ 0 & T & 1 & -X_{10}T^2/2 + X_{10}^3/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{12}(T, 0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2U/P_2 & -2Ut_s/P_2 & 0 \\ 0 & 2UT/P_2 & -2Ut_sT/P_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_{21}(T, 0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -UP_2X_{10}T/2 \\ 0 & 0 & 0 & 0 \\ 0 & -UP_2X_{10}T & 0 & C \end{bmatrix}$$

where $C = P_2X_{10}t_s(T - t_s)/2 + UP_2(T - t_s)^4/6 + t_s^3/6$

$$\Phi_{22}(T, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -UX_{10}^2/2 & -X_{10}^3/6 & 1 \end{bmatrix}$$

To obtain the gain matrix M, we now make use of the boundary conditions (25) and (26) which result in

$$\xi_1(T) = -\dot{X}_1(T)\, dT = U\, dT$$

$$\xi_2(T) = -\dot{X}_2(T)\, dT = 0$$

$$\psi_0(T) = -\dot{P}_0(T)\, dT = 0$$

$$\psi_3(T) = -\dot{P}_3(T)\, dT = 0$$

$$P_2(T)\xi_1(T) - U\psi_1(T) = 0$$

respectively.

Using these relations with the fundamental matrix $\Phi(T, 0)$ previously calculated, and then eliminating T and t_s by use of (1-9) we find the relations between ψ_{i0} and ξ_{i0}:

$$\begin{bmatrix} \psi_{10} \\ \psi_{20} \\ \psi_{30} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{bmatrix} \qquad (1-18)$$

where
$$m_{11} = P_2^3 X_{20}$$
$$m_{12} = -P_2^3 X_{10}/2$$
$$m_{13} = P_2^3 X_{10}^4/6 - P_2 X_{10}^2 + X_{10}$$
$$m_{21} = -P_2^3 X_{10}/2$$
$$m_{22} = -UP_2^3/2$$
$$m_{23} = -UP_2^3 X_{10}^3/6$$
$$m_{31} = P_2^3 X_{10}^4/6 - P_2 X_{10}^2 + X_{10}$$
$$m_{32} = -UP_2^3 X_{10}^3/6$$
$$m_{33} = U(P_2^3 X_{10}^6/18 - P_2 X_{10}^4/2 + 2X_{10}^3/3 - P_2^3/3)$$

The 3×3 matrix above gives the elements $m_{ij}(0)$, $i, j = 1, 2, 3$. (The $m_{0i} = m_{i0}$ terms are obviously zero and not needed.) Since X_{10} and X_{20} are arbitrary, we can drop the subscripts 0 everywhere in (1-18) and thus obtain the relation between $\psi = \psi(t)$, $\xi = \xi(t)$ for and $X = X(t)$. Consequently the last three rows and columns of $M(X)$ are given by the matrix in (1-18).

Only the element m_{13} of the above matrix is needed for the quasi-optimum control law (1-13), which thus becomes

$$u = \operatorname{sgn}\,[P_1 + (\tfrac{1}{6}\, P_2^3 X_1^4 - P_2 X_1^2 + X_1)\, a]\ , \quad a = x_3 \tag{1-19}$$

where P_1 and P_2 given by (1-11). The approximate switching curve is obtained by setting the bracketed expression in (1-19) equal to zero. The resulting switching surface is the curve labeled QUASI-OPTIMUM in Fig. 1-1, for $a = 0.3$.

Comparison with Exact Solution - The exact problem in this case admits an analytic solution which can be compared with the control laws for the simplified problem and the quasi-optimum control law (1-19). The exact solution is obtained as follows. The canonical equations (1-1), (1-4) and (1-5) can be integrated explicitly over the interval $[t, T]$. Equating the results to the corresponding terminal conditions results in

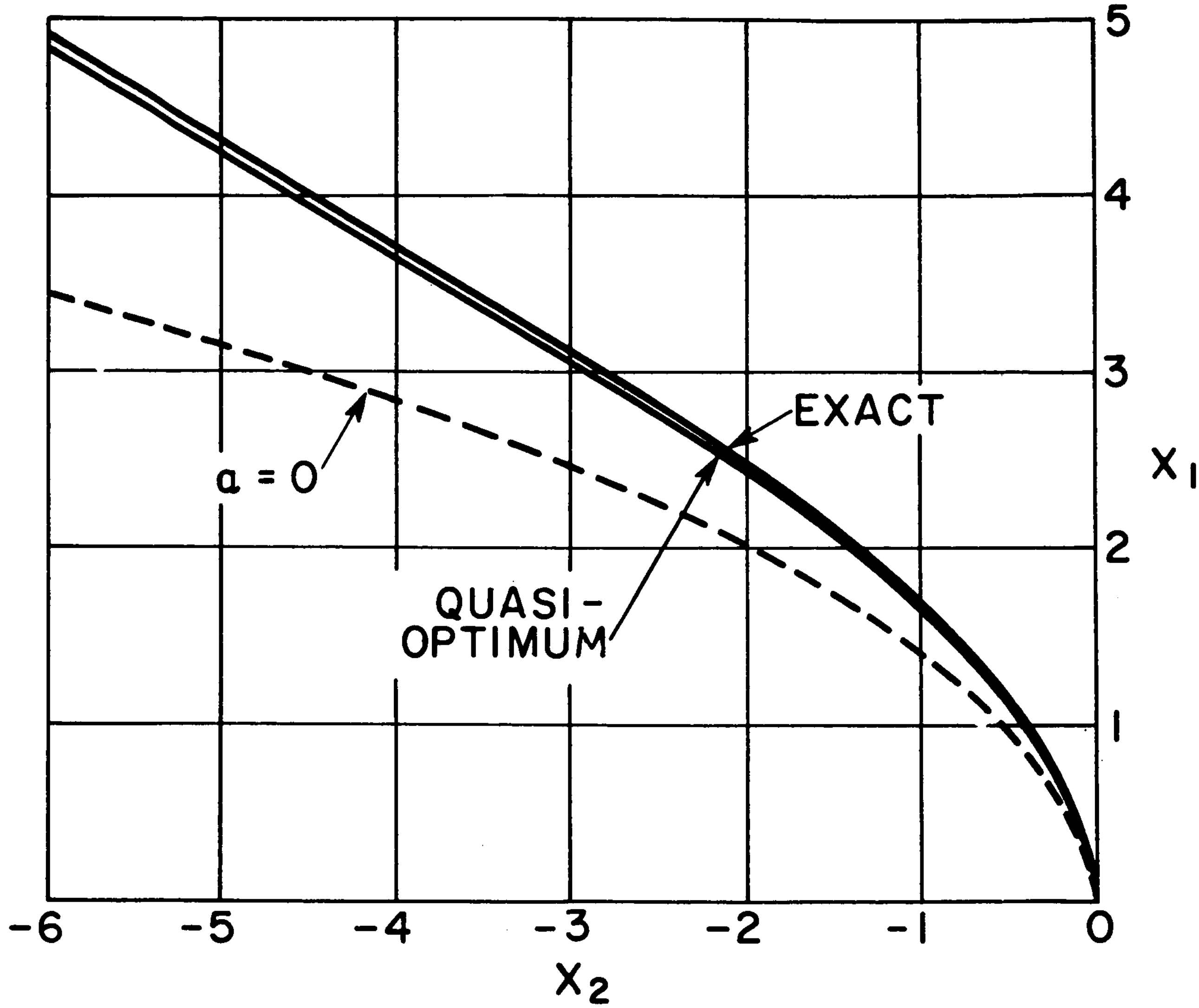


FIGURE I-1
COMPARISON OF EXACT AND
QUASI-OPTIMUM SWITCH CURVES
FOR BUSHAW PROBLEM
a = 0.3

$$x_0(T) = \tau$$

$$0 = x_1(T) = x_1 + \frac{U}{x_3}\,(2\,e^{x_3 t_s} - e^{-a\tau} - 1)$$

$$0 = x_2(T) = x_2 + \frac{1-e^{-x_3\tau}}{x_3}\,x_1 + \frac{U}{x_3}\,[\frac{1}{x_3} - \tau + 2t_s + e^{-x_3\tau}(\frac{1-2e^{x_3 t_s}}{x_3})]$$

$$a = x_3(T) = x_3(= \text{constant})$$

$$-1 = p_0(T) = p_0(= \text{constant})$$

$$p_1(T) = e^{x_3\tau}\,[p_1 - p_2(\frac{1-e^{-x_3\tau}}{x_3})]$$

$$p_2(T) = p_2(= \text{constant})$$

where
$$\tau = T - t$$

$$t_s = \frac{1}{x_3}\log\,(\frac{p_2}{1 - p_1 x_3})$$

$$U = \text{sgn}\; p_1 = \pm\; 1$$

Using these relations and the Hamiltonian (1-3) we obtain

$$\tau = -U(x_1 + a x_2) + \frac{2\log S}{a}$$

$$p_1 = U - \frac{a x_1 (S-2)}{(aUx_1 - 1)(S - 1)}$$

$$p_2 = \frac{aU}{s-1}$$

$$p_3 = Ux_2 + \frac{2\log S}{a^2} - (\frac{x_1^2}{aUx_1 - 1} + 2Ux_2)\,\frac{S-2}{S-1}$$

where
$$S = 1 + [\,(aUx_1 - 1)\,e^{aU(x_1 + a x_2)} + 1]^{\frac{1}{2}}$$

The exact switch surface is given by setting $p_2 = 0$ and is found to be

$$x_2 = -\frac{x_1}{a} + \frac{U}{a^2} \log (1 + aUx_1) \tag{1-20}$$

and is the curve labeled EXACT in Figure 1-1. It is seen that the exact and approximate curves differ by less than one percent for $x_1 < 4$, whereas the curve which would be obtained for $a = 0$ differs considerably from the optimum.

It is also of interest that the expression obtained by expanding the exact switching curve (1-20) in a power series in a, namely

$$x_2 = -\frac{1}{2}Ux_1^2 + \frac{1}{3}ax_1^3 - \frac{1}{4}Ua^2x_1^4 + \dots,$$

and retaining only the term that is linear is a very poor approximation for $ax_1^2 > 1$ or, in this case, for $x_1 > 1.73$. In fact the curve is not even single-valued.

2.2 MINIMUM-TIME, BOUNDED ACCELERATION RENDEZVOUS IN FREE SPACE

The second example chosen to illustrate the quasi- optimum control technique was that of rendezvous in free-space (zero-g) in minimum time.

Exact Problem - The motion of the vehicle relative to the target is described in a target-referenced polar coordinate system (see Figure 2-1) as follows

$$
\begin{aligned}
d^2r/dt^2 - r\, d\varphi/dt &= f_r/m \\
r\, d^2\varphi/dt^2 + 2\,(dr/dt)(d\varphi/dt) &= f_\varphi/m
\end{aligned}
\qquad (2\text{-}1)
$$

Define a new set of variables as follows

$$x_0 = t\,,\; x_1 = (dr/dt)/a\,,\; x_2 = r/a\,,\; x_3 = r\,(d\varphi/dt)/a$$

where the acceleration $a = f/m$.

In terms of these new variables, the equations of motion become

$$
\begin{aligned}
\dot{x}_0 &= 1 \\
\dot{x}_1 &= x_3^2/x_2 + u_1 \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= -x_1 x_3/x_2 + u_2\,,
\end{aligned}
\qquad (2\text{-}2)
$$

where

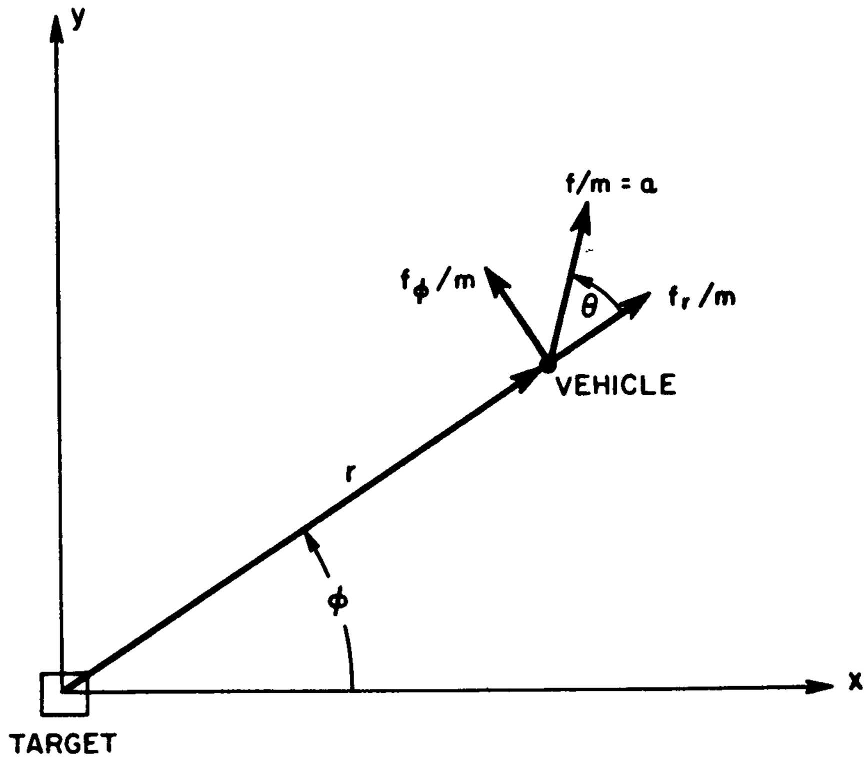
$$u_1 = \cos\theta\,,\; u_2 = \sin\theta \qquad (2\text{-}3)$$

The problem is then to minimize $x_0(T)$ subject to the constraint

$$u_1^2 + u_2^2 = 1 \qquad (2\text{-}4)$$

The Hamiltonian for this problem is

$$h = p_0 + p_1(x_3^2/x_2 + u_1) + p_2 x_1 + p_3(-x_1 x_3/x_2 + u_2) \qquad (2\text{-}5)$$



TARGET REFERENCED
POLAR CO-ORDINATE SYSTEM
FIGURE 2-1

Maximization of h with respect to u_1 and u_2 subject to (2-4) results in the following steering law

$$u_1 = p_1/(p_1^2 + p_3^2)^{\frac{1}{2}}$$

and
$$u_2 = p_3/(p_1^2 + p_3^2)^{\frac{1}{2}} \tag{2-6}$$

Using these values of u_1 and u_2, along with the condition that $p_0 \equiv -1$ in h, yields

$$h = -1 + p_1 x_3^2/x_2 + p_2 x_1 - p_3 x_1 x_3/x_2 + (p_1^2 + p_3^2)^{\frac{1}{2}} \tag{2-7}$$

Simplified Problem - Suppose that the initial tangential velocity is zero. Then the optimum solution evidently is to apply the acceleration along the initial radius vector pointing either toward or away from the origin in accordance with the well-known solution for the one-dimensional process $d^2x_2/dt^2 = u_1$ (Bushaw's Problem).

If the initial tangential velocity is suitably small, it is reasonable to use the solution of the one-dimensional problem as the basis for an approximate solution to the two-dimensional problem. Thus we select as the state of the simplified process

$$X = \{x_0, x_1, x_2, 0\} \tag{2-8}$$

Then
$$\xi = \{\xi_0, 0, 0, x_3\}$$

(Note that ξ_0 is the approximate change in performance due to the simplification).

For the one-dimensional problem the Hamiltonian is

$$H = P_0 + P_1 U + P_2 X_1, \tag{2-9}$$

where $P = [P_0, P_1, P_2, 0]$ is the adjoint vector in the simplified problem. The maximum principle applied to (2-9) gives

$$U = \text{sgn } P_1(t) \tag{2-10}$$

The adjoint equations for the simplified problem are determined by using (2-9) in the canonical equations $dP_i/dt = -\partial H/\partial X_i$ and are

$$
\begin{aligned}
\dot{P}_0 &= 0 \\
\dot{P}_1 &= -P_2 \\
\dot{P}_2 &= 0
\end{aligned}
\tag{2-11}
$$

Integrating (2-11) results in

$$
\begin{aligned}
P_0 &= -1 \\
P_1 &= P_{10} - P_{20}t \\
P_2 &= P_{20},
\end{aligned}
$$

where P_{10} and P_{20} are constants and t is time measured from the (arbitrary) starting instant. The optimum control law for the simplified system (2-10) becomes

$$
U_1 = \operatorname{sgn}(P_{10} - P_{20}t)
$$

$$
U_1 = \begin{cases} U, & t < P_{10}/P_{20} = t_s \\ -U, & t > P_{10}/P_{20} \end{cases}
\tag{2-12}
$$

where $U = \operatorname{sgn}(P_{10}/P_{20}) = \pm 1$

Substituting (2-12) into (2-2) with $x_3 \equiv 0$ and integrating to the terminal time T, results in expressions for $x_1(T)$ and $x_2(T)$ as functions of T, t_s and the initial conditions. Solving simultaneously for T and t_s and utilizing these along with (2-9) results in the following expressions for the initial adjoint variables in terms of the initial state variables.

$$
P_{10} = U - X_{10}/(X_{10}^2/2 - UX_{20})^{\frac{1}{2}}
\tag{2-13a}
$$

$$
P_{20} = U/(X_{10}^2/2 - UX_{20})^{\frac{1}{2}}
\tag{2-13b}
$$

Since the initial state is arbitrary, dropping the subscript zero in (2-13) results in the general relations for the adjoint variables in terms of the state variables. Substitution of (2-13a) into (2-10) leads to the well-known control law for the simplified problem.

If the initial tangential velocity is not absolutely zero, however, this control law is unsatisfactory for the original problem because no tangential acceleration is ever produced. As a result the initial angular momentum is conserved, and as the radial distance decreases the tangential velocity increases until the vehicle either orbits the origin or escapes entirely. Satisfactory performance can be achieved only by use of a tangential component of acceleration.

Quasi-Optimum Control Law - In the quasi-optimum control law the radial and tangential components of the normalized acceleration are given by (2-6), in which approximate values of p_1 and p_3 are used. These approximations are given by

$$p_i = P_i + \sum_{j=0}^{3} m_{ij}\xi_i \qquad i = 1, 3 \tag{2-14}$$

From (2-8) however, $\xi_1 = \xi_2 = 0$ and $\xi_3 = x_3$; hence (2-14) becomes

$$p_1 = P_1 + m_{10}\xi_0 + m_{13}\xi_3 \tag{2-15}$$

$$p_3 = m_{30}\xi_0 + m_{33}\xi_3 \tag{2-16}$$

Thus only m_{10}, m_{30}, m_{13}, and m_{33} in the matrix M are needed. These are calculated with the aid of (19). The coefficient matrices $H_{XX}, \ldots, H_{PP}$ appearing thereon are found by performing the required partial differentiations on the Hamiltonian for the complete problem, given by (2-7), and evaluating the result at $x = X$, i.e., for $x_3 \equiv 0$. The results are

$$H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2P_1/X_2 \end{bmatrix} \qquad H_{PP} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U/P_1 \end{bmatrix}$$

$$H_{XP} = H'_{PX} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -X_1/X_2 \end{bmatrix}$$

The result of substituting these matrices into the auxiliary system (20) expressed in component form, is

$$\dot{\xi}_0 = 0 \tag{2-17a}$$

$$\dot{\xi}_1 = 0 \tag{2-17b}$$

$$\dot{\xi}_2 = \xi_1 \tag{2-17c}$$

$$\dot{\xi}_3 = -(X_1/X_2)\xi_3 + (U/P_1)\psi_3 \tag{2-17d}$$

$$\dot{\psi}_0 = 0 \tag{2-17e}$$

$$\dot{\psi}_1 = \psi_2 \tag{2-17f}$$

$$\dot{\psi}_2 = 0 \tag{2-17g}$$

$$\dot{\psi}_3 = -(2\,P_1/X_2)\xi_3 + (X_1/X_2)\psi_3 \tag{2-17h}$$

Note that the equations for ψ_3 and ξ_3 are uncoupled from the others (which can be integrated by quadratures). Thus

$$\xi_0(T) = \xi_0(0) = \text{const.} \tag{2-18a}$$

$$\xi_1(T) = \xi_1(0) = \text{const.} \tag{2-18b}$$

$$\xi_2(T) = \xi_1(t)\tau + \xi_2(t) \tag{2-18c}$$

$$\xi_3(T) = a(t)\,\xi_3(t) + b(t)\,\psi_3(t) \tag{2-18d}$$

$$\psi_0(T) = \psi_0(t) = \text{const.} \tag{2-18e}$$

$$\psi_1(T) = -\psi_2(t)\tau + \psi_1(t) \tag{2-18f}$$

$$\psi_2(T) = \psi_2(t) = \text{const.} \tag{2-18g}$$

$$\psi_3(T) = c(t)\,\xi_3(t) + b(t)\,\psi_3(t) \tag{2-18h}$$

where $\qquad \tau = T - t$

The above are subject to boundary conditions (25) and (26) which, for this problem, become

$$\xi_1(T) = -\dot{X}_1(T)\,dT = (\mathrm{sgn}\ P_1)\,dT \tag{2-19a}$$

$$\xi_2(T) = -\dot{X}_2(T)\,dT = 0 \tag{2-19b}$$

$$\psi_0(T) = -\dot{P}_0(T)\,dT = 0 \tag{2-19c}$$

$$\psi_3(T) = -\dot{P}_3(T)\,dT = 0 \tag{2-19d}$$

and

$$-P_2(T)\xi_1(T) = (\mathrm{sgn}\ P_1)\psi_1(T) \tag{2-19e}$$

From (2-18e) and (2-19c) we obtain $\psi_0 \equiv 0$ and hence $m_{0i} = m_{j0} = 0$. From (2-18h) and (2-19d), $\psi_3(t)$ is only a function of $\xi_3(t)$, therefore $m_{31} = 0$. By symmetry of M, $m_{13} = m_{23} = 0$. Thus the only non-zero component in (2-15) and (2-16) is m_{33}. Thus (2-15) and (2-16) become

$$P_1 = P_1 \tag{2-20a}$$

and

$$P_3 = m_{33}X_3 \tag{2-20b}$$

The remaining coefficient $m_{33}(t)$ is given by (2-18h) and (2-19d): $m_{33}(t) = \psi_3(t)/\xi_3(t) = -c(t)/b(t)$. To obtain $c(t)$ and $b(t)$, however, the time-varying second order system (2-17d) and (2-17h) must be solved. We were unable to solve this system and accordingly obtained a scalar Riccati equation for m_{33} through use of (2-20b):

$$-\,dm_{33}/dt = -(2X_1/X_2)m_{33} + (U/P_1)m_{33}^2 + 2P_1/X_2 \tag{2-21}$$

Since this equation is equally intractable, an approximate solution was obtained by assuming $dm_{33}/dt = 0$; i.e.,

$$m_{33} = (P_1U/X_2)\,[X_1 \pm (X_1^2 - 2X_2U)^{\frac{1}{2}}] \tag{2-22}$$

where P_1 and U are given by (2-13a) and (2-12), respectively, (with the zero subscript omitted). The choice of the plus or minus sign of the square root term in (2-22) was resolved by recognizing that in order to get the proper acceleration direction the sign of the square root term must be negative. (e.g., for $X_1 = 0$ and $X_2 = 1.0$, $U = -1.0$; the negative square root then gives the proper retarding effect on x_3.) Hence, use of (2-22) with the negative sign in (2-20b) and (2-20a) in (2-6) results in the following quasi-optimum control law.

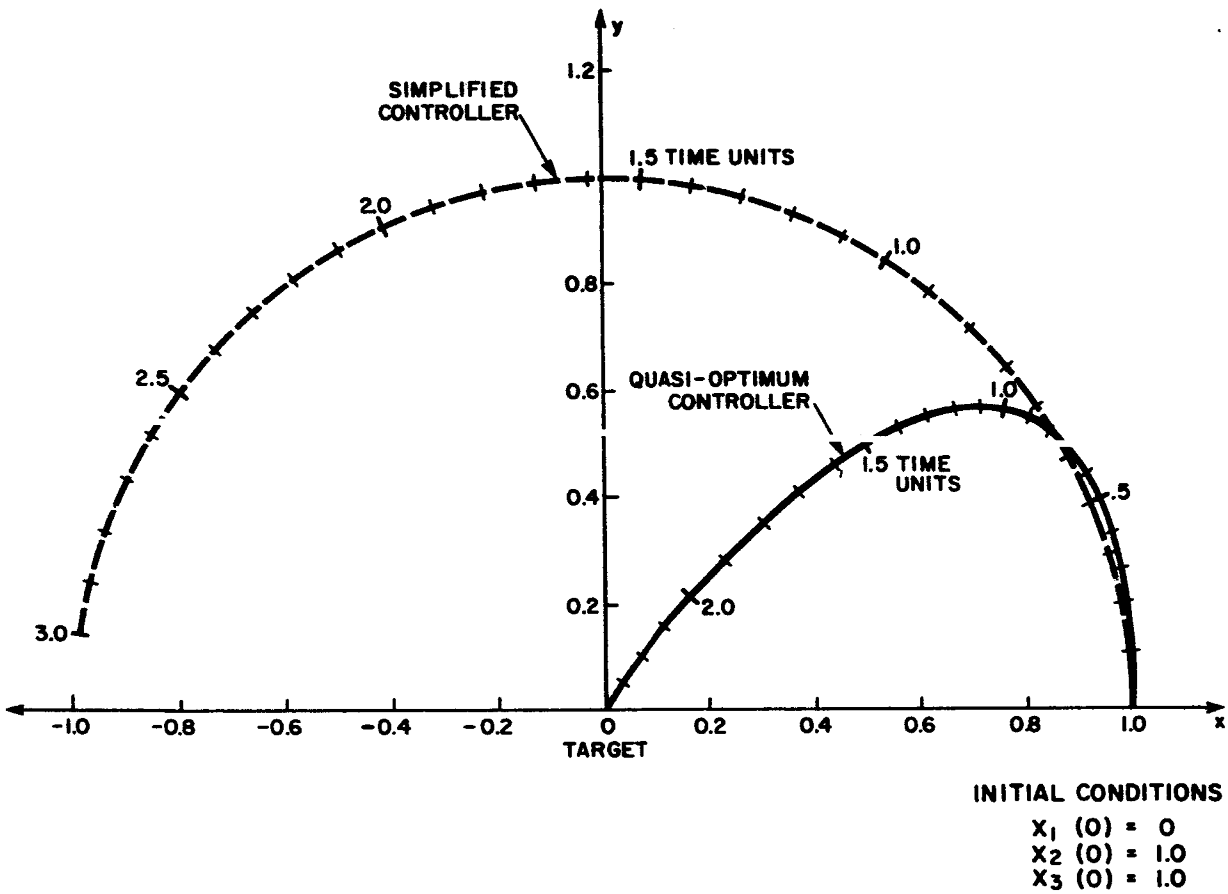
$$u_1 = P_1/\,\{P_1^2 + [(P_2U/X_2)(X_1 - \sqrt{2}\,UP_2)]^2\,x_3^2\}^{\frac{1}{2}} \tag{2-23}$$

$$u_2 = (P_1 U/X_2)(X_1 - \sqrt{2UP_2})\, x_3 / \{P_1^2 + [(P_2 U/X_2)(X_1 - \sqrt{2UP_2})]^2 x_3^2\}^{\frac{1}{2}} \qquad (2\text{-}24)$$

(The square root term of (2-22) has been eliminated by use of (2-13b)).

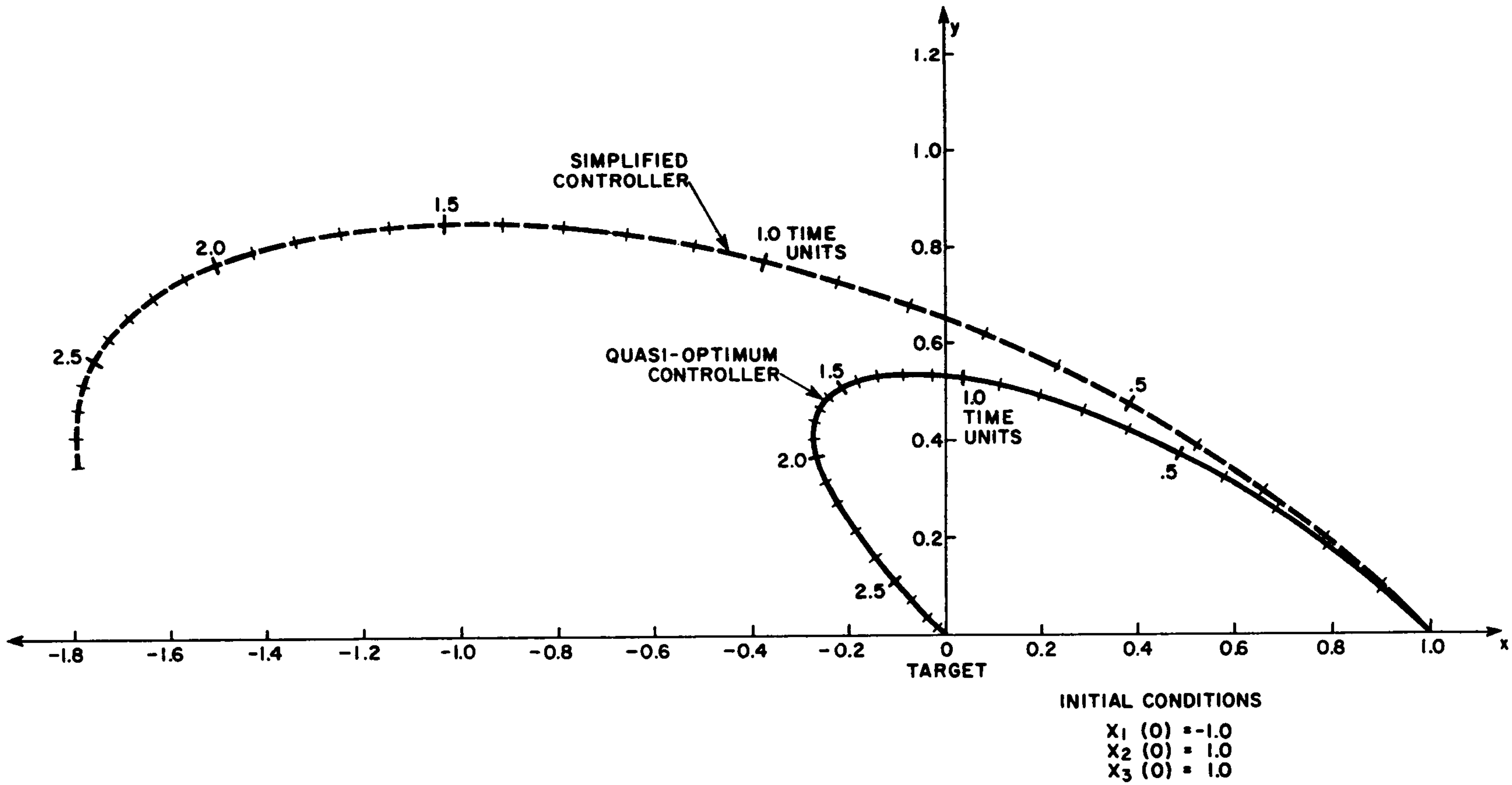
<u>Performance with Quasi-Optimum Control Law</u> - The performance of the control system using the quasi-optimum control law (2-23) and (2-24) was simulated with the aid of a digital computer. For purposes of comparison, the performance using the control law for the simplified problem (2-12) was also simulated. Trajectories for three initial conditions are shown in Figures 2-2, 2-3, and 2-4. The unsatisfactory performance with the controller for the simplified problem is evident. By using the quasi-optimum controller, however, the rendezvous is achieved for all practical purposes: the vehicle is steered very close to the target and the velocity is simultaneously reduced very nearly to zero.

It should be noted that this good performance was achieved even though the initial tangential velocity was not small, as is clear from the size of the excursion from the initial radius vector.



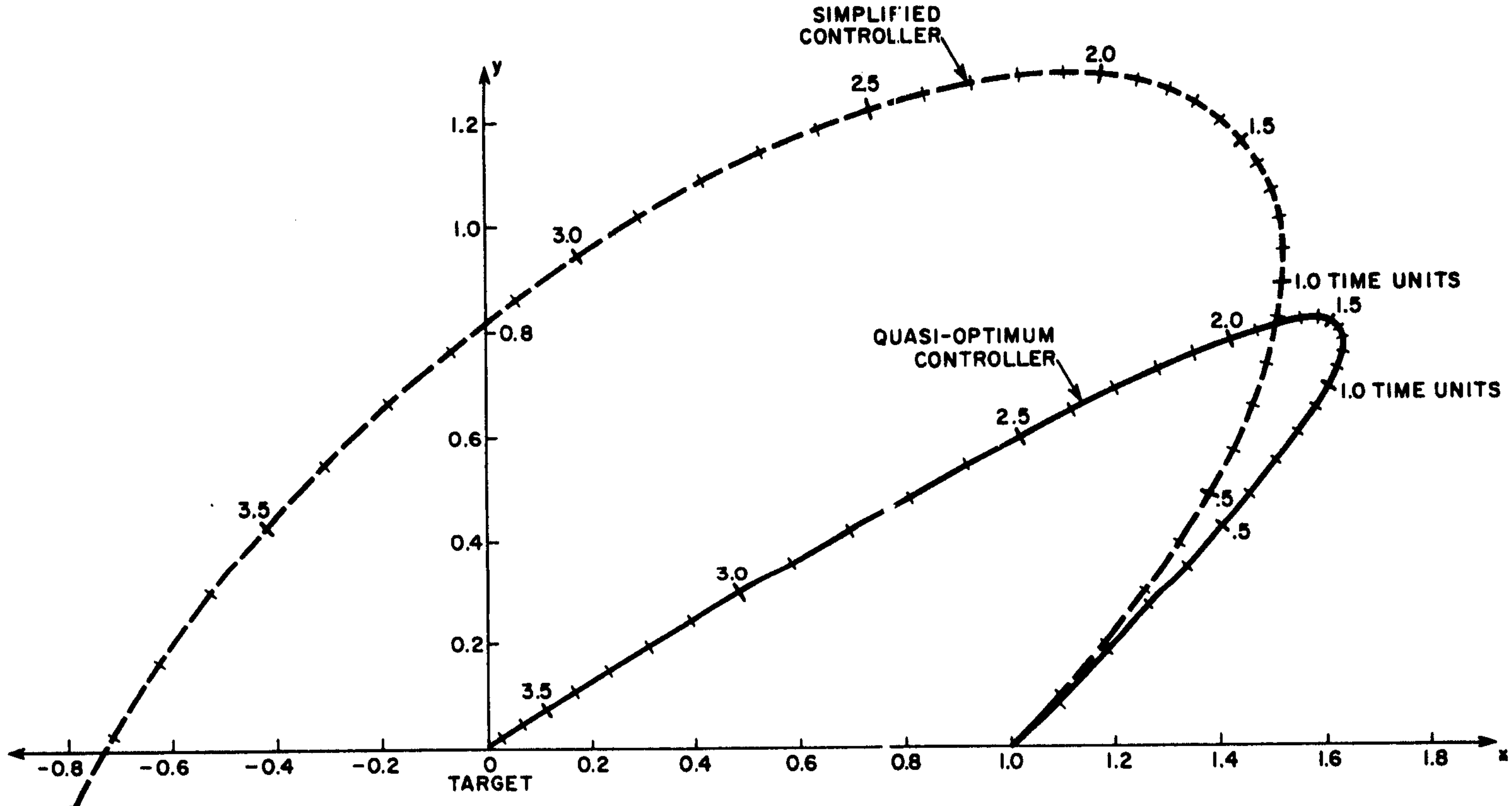
COMPARISON OF SIMPLIFIED AND QUASI-OPTIMUM CONTROLLED TRAJECTORIES

FIGURE 2-2



COMPARISON OF SIMPLIFIED AND QUASI-OPTIMUM CONTROLLED TRAJECTORIES

FIGURE 2-3



INITIAL CONDITIONS

X_1 (0) = 1.0
X_2 (0) = 1.0
X_3 (0) = 1.0

COMPARISON OF SIMPLIFIED AN(I QUASI-OPTIMUM CONTROLLED TRAJEC''ORIES

FIGURE 2-4

2.3 FLEXIBLE BOOSTER ATTITUDE CONTROL

The problem of controlling the attitude of a flexible booster was chosen as another illustration of the quasi-optimum control technique. For simplicity we have selected a model with only a single bending mode present and with negligible actuator dynamics but the technique can readily be extended to include actuator dynamics and more bending modes. The basic assumption for the simplified problem is that the flexible booster can be represented as a rigid body. A first-order correction is then applied to account for the flexibility.

Exact Problem - The dynamic model for the exact problem is taken to be two rigid homogeneous sections, each having its own physical and aerodynamic characteristics linked by a torsional spring. (See Figure 3-la).

The linearized equations of motion for section 1 of the dynamic model are given by

$$m_1\dot{v}_{1x} = T - h + f\theta_1 - D_1 - L_1\theta_1 \tag{3-1}$$

$$m_1\dot{v}_{1y} = T(\theta_1 - \delta) - h\theta_1 - f + L_1 - D_1\gamma_1 \tag{3-2}$$

$$J_1\ddot{\theta}_1 = T\delta g_1 - f(\ell_1 - g_1) + C_\alpha(\theta_2 - \theta_1) + (L_1 - D_1\gamma_1 + D_1\theta_1)S_1 \tag{3-3}$$

and for section 2 are given by

$$m_2\dot{v}_{2x} = h - f\theta_1 - D_2 - L_2\theta_2 \tag{3-4}$$

$$m_2\dot{v}_{2y} = h\theta_1 + f + L_2 - D_2\gamma_2 \tag{3-5}$$

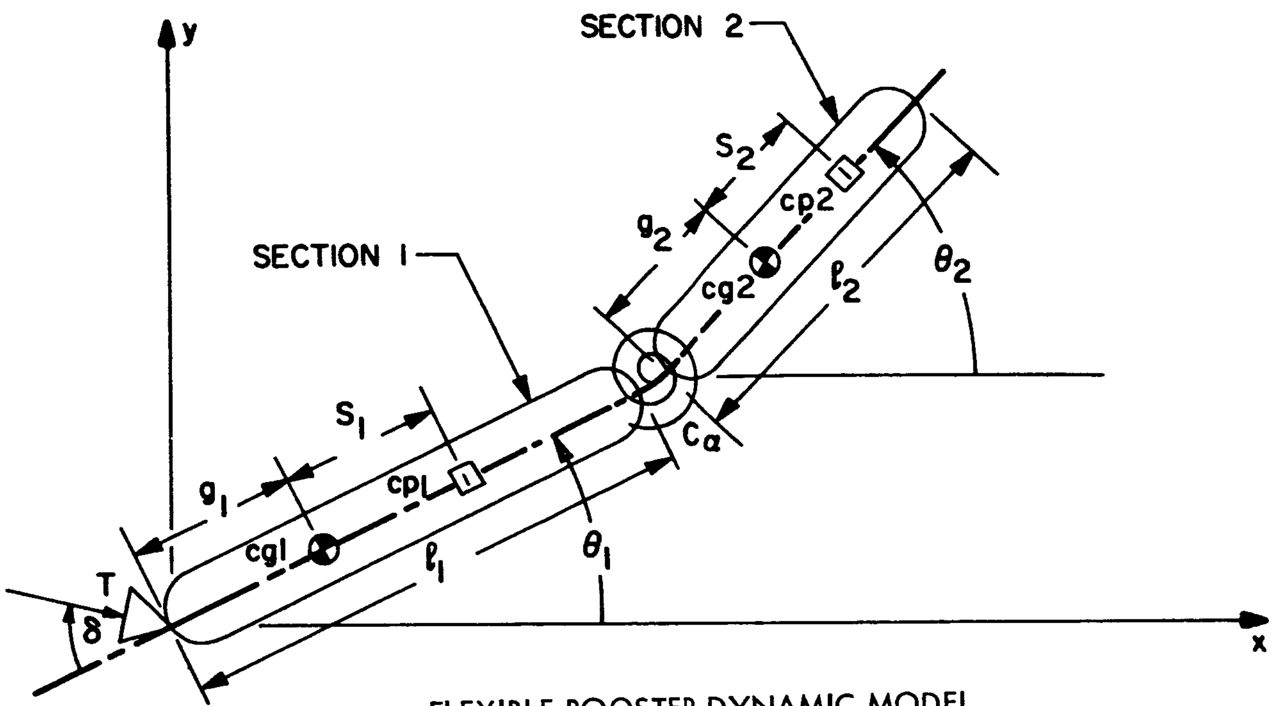
$$J_1\dot{\theta}_2 = -fg_2 + (hg_2 - C_\alpha)(\theta_2 - \theta_1) + (L_2 - D_2\theta_2)S_2 \tag{3-6}$$

where the above parameters are illustrated in the free body diagram of the dynamic model (See Figure 3-lb). Due to the interconnection there are two position constraints on the booster

$$x_1 + \ell_1 - g_1 = x_2 - g_2$$

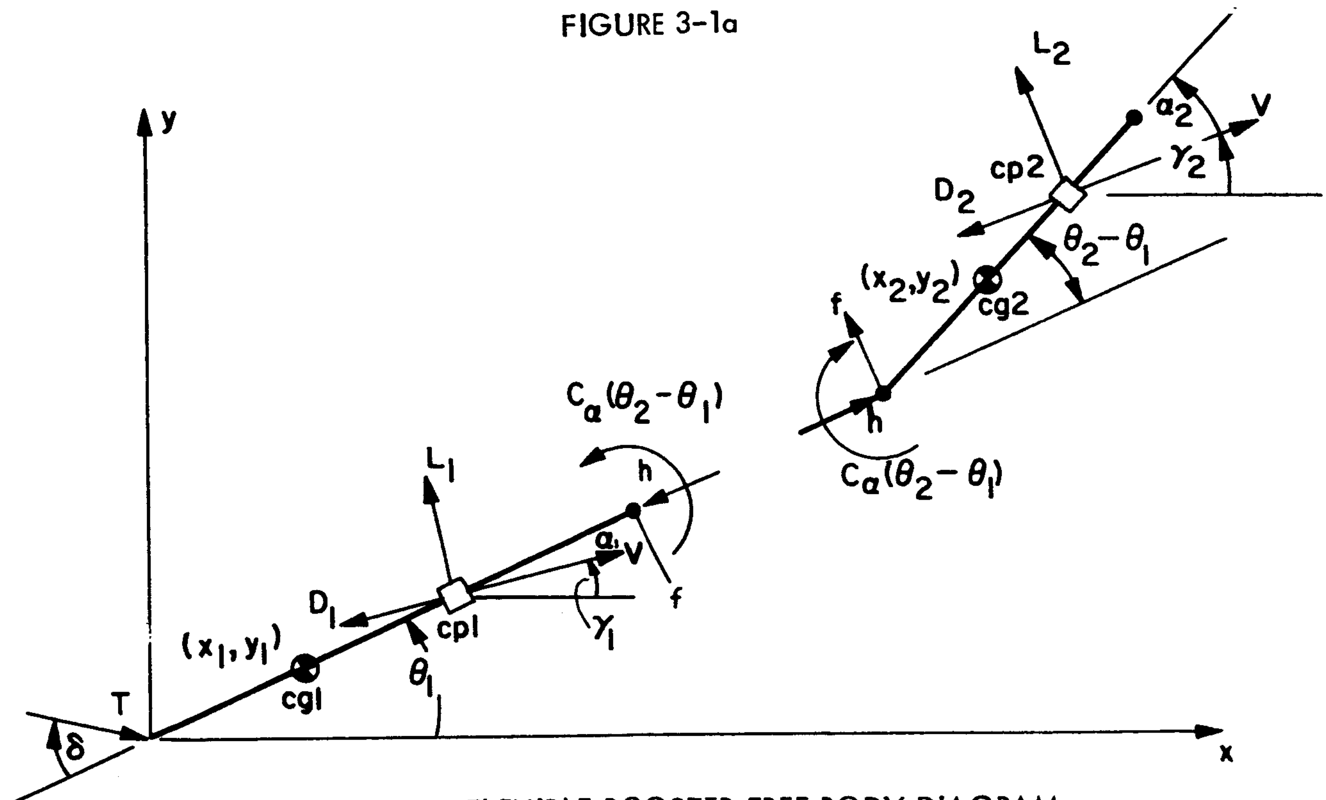
and $$y_1 + (\ell_1 - g_1)\theta_1 = y_2 - g_2\theta$$

in the x any y directions respectively.



FLEXIBLE BOOSTER DYNAMIC MODEL

FIGURE 3-1a



FLEXIBLE BOOSTER FREE BODY DIAGRAM

FIGURE 3-1b

Differentiating the above two equations twice with respect to time yields the following two kinematic constraints on the accelerations:

$$\dot{v}_{1x} = \dot{v}_{2x} \tag{3-7}$$

$$\dot{v}_{2y} - \dot{v}_{1y} = (\ell_1 - g_1)\ddot{\theta}_1 + g_2\ddot{\theta}_2 \tag{3-8}$$

Using (3-1) and (3-4) in (3-7) and solving for h yields

$$h = f\theta_1 + m_1 m_2 [T/m_1 - (D_1 + L_1\theta_1)/m_1 + (D_2 + L_2\theta_2)/m_2]/(m_1 + m_2) \tag{3-9}$$

Using (3-2), (3-5) and (3-9) in (3-8) and solving for f yields

$$f = m_1 m_2 [- T\delta/m_1 + f_{\alpha 1}/m_1 - f_{\alpha 2}/m_2 + (\ell_1 - g_1)\theta_1 + g_2\ddot{\theta}_2]/(m_1 + m_2) \tag{3-10}$$

where $$f_{\alpha i} \equiv L_i - D_i\gamma_i + D_i\theta_i \qquad i = 1, 2 \tag{3-11}$$

Utilizing (3-9) and (3-10) in (3-3) and (3-6) and solving simultaneously for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ results in

$$\begin{aligned}\ddot{\theta}_1 = [1/(AB - C^2)]\{&T\delta[B(m_1 g_1 + m_2\ell_1) - Cm_2 g_2]/M + f_{\alpha 1}[Cm_2 g_2/M + B[S_1 - m_2(\ell_1 - g_1)/M]] \\ &+ f_{\alpha 2}[Bm_1(\ell_1 - g_1)/M - C[S_2 + m_1 g_2/M]] \\ &+ (\theta_2 - \theta_1)[C_\alpha(B + C) - g_2 C(Tm_2 - D_1 m_2 + D_1 m_1)/M]\}\end{aligned} \tag{3-12}$$

and

$$\begin{aligned}\ddot{\theta}_2 = [1/AB - C^2)]\{&T\delta[Am_2 g_2 - C(m_1 g_1 + m_2\ell 1)]/M + f_{\alpha 1}[- Am_2 g_2/M - C[S_1 - m_2(\ell_1 - g_1)/M]] \\ &+ f_{\alpha 2}[- Cm_1(\ell_1 - g_1)/M + A[S_2 + m_1 g_2/M]] \\ &+ (\theta_2 - \theta_1)[- C_\alpha(A + C) + Ag_2(Tm_2 - D_1 m_2 + D_2 m_1)/M]\}\end{aligned} \tag{3-13}$$

where

$$A = J_1 + m_1 m_2(\ell_1 - g_1)^2/M$$

$$B = J_2 + m_1 m_2 g_2^2/M$$

$$C = m_1 m_2(\ell_1 - g_1)g_2/M$$

and $$M = m_1 + m_2$$

• Utilizing the following standard aerodynamic terminology for the terms in $f_{\alpha i}$ given in (3-11):

$$L_i = C_{L\alpha i} qS'\alpha = \mu_{Li}\alpha$$

$$D = C_D qS'$$

where

$$q = \rho V^2/2$$

$$\alpha = \theta - \gamma$$

$$\gamma = v_y/V$$

$$|V| \cong \text{constant}$$

results in

$$f_{\alpha i} = \mu_{fi}(\theta_i - v_{yi}/V) \qquad i = 1, 2 \tag{3-14}$$

where

$$\mu_{fi} = q_i S'_i (C_{L\alpha i} + C_{Di})$$

It is desirable to write the equations of motion in a form that will facilitate the extraction of the simplified problem from the exact problem. For this purpose, we define a new set of variables:

$$\theta_{av} = [(A + C)\theta_1 + (B + C)\theta_2]/(A + B + 2C)$$

$$\varphi = (\theta_2 - \theta_1) \tag{3-15}$$

$$v_{av} = [m_1 v_{1y} + m_2 v_{2y} - m_1 m_2(\ell_1 - g_1 + g_2)(\dot{\theta}_2 - \dot{\theta}_1)/M]/M$$

In terms of these variables the equations of motion can be written in the vector matrix form

$$\dot{x} = Ax + \beta u \tag{3-16}$$

where

$$x = \begin{bmatrix} v_{av} \\ y_{av} \\ \dot{\theta}_{av} \\ \theta_{av} \\ \dot{\varphi} \\ \varphi \end{bmatrix} \qquad \beta = \begin{bmatrix} -a_T \\ 0 \\ \mu_c \\ 0 \\ q \\ 0 \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} -d_1 & 0 & 0 & h & -\ell & n \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\mu_\alpha/V & 0 & 0 & \mu_\alpha & b & c \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -w/V & 0 & 0 & w & d_6 & -\omega^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The non-zero terms in $\mathcal{A}$ and β are defined by:

$$\mu_\alpha = \mu_f S/J_{TOT}$$

$$\mu_c = Tg/J_{TOT}$$

$$d_1 = \mu_f/MV$$

$$h = (T + \mu_L)/M$$

$$a_T = T/M$$

$$b = m_1 m_2(\ell_1 - g_1 + g_2)\mu_f S/M^2 V J_{TOT}$$

$$c = [(A + C)\mu_{f2}\bar{S}_2 - (B + C)\mu_{f1}\bar{S}_1]/J^2_{TOT} + g_2(Tm_2 - D_1 m_2 + D_2 m_1)/MJ_{TOT}$$

$$\ell = m_1 m_2(\ell_1 - g_1 + g_2)\mu_f/M^3 V$$

$$n = [(A + C)\mu_{L2} - (B + C)(\mu_{L1} + T)]/MJ_{TOT}$$

$$w = \{-\mu_{f1}[m_2 g_2 J_{TOT}/M + \bar{S}_1(B + C)] + \mu_{f2}[-m_1(\ell_1 - g_1)J_{TOT}/M + \bar{S}_2(A + C)]\}/(AB - C^2)$$

$$d_6 = m_1 m_2(\ell_1 - g_1 + g_2)\{\mu_{f1}[m_2 g_2 J_{TOT}/M + \bar{S}_1(B + C)]$$

$$- \mu_{f2}[- m_1(\ell_1 - g_1)J_{TOT}/M + \bar{S}_2(A + C)]\}/M^2 V(AB - C^2)$$

$$-\omega^2 = \{\mu_{f1}(B + C)[m_2 g_2 J_{TOT}/M + \bar{S}_1(B + C)]/J_{TOT}$$

$$+ \mu_{f2}(A + C)[- m_1(\ell_1 - g_1)J_{TOT}/M + \bar{S}_2(A + C)]/J_{TOT}$$

$$- C_\alpha J_{TOT} + g_2(A + C)(Tm_2 - D_1 m_2 + D_2 m_1)/M\}/(AB - C^2)$$

$$q = T[m_2 g_2(A + C) - (m_1 g_1 + m_2 \ell_1)(B + C)]/M(AB - C^2)$$

where

$$\bar{S}_1 = S_1 - m_2(\ell_1 - g_1 + g_2)/M$$

$$\bar{S}_2 = S_2 + m_1(\ell_1 - g_1 + g_2)/M$$

$$J_{TOT} = A + B + 2C$$

$$g = [m_1 g_1 + m_2(\ell_1 + g_2)]/M$$

$$S = [\mu_{f1}\bar{S}_1 + \mu_{f2}\bar{S}_2]/\mu_f$$

$$\mu_f = \mu_{f1} + \mu_{f2}$$

$$\mu_L = \mu_{L1} + \mu_{L2}$$

The block diagram for (3-16) is shown in Figure 3-2. The heavy lines denote the rigid-body dynamics; the contribution to the motion due to the flexibility is indicated by the lighter-weight lines.

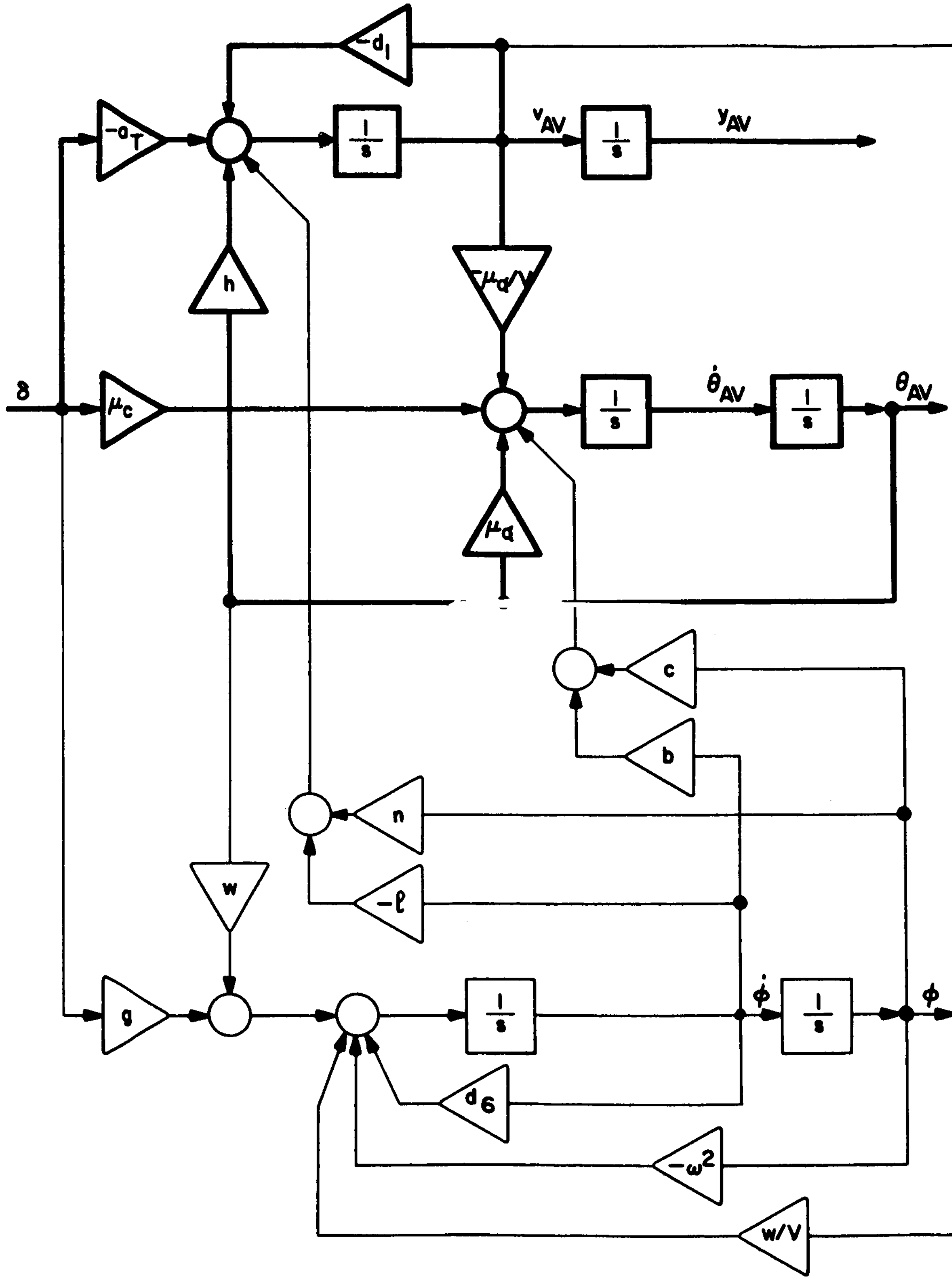
The controller to be designed will minimize the following performance index

$$V_0(T) = x_0(T) = y_{av}^2(T)/2 + (K^2/2)\int_t^T M_{av}^2\, d\tau \tag{3-17}$$

where T = terminal time

K^2 = weighting factor

M_{av} = total moment about the mean center of gravity



BLOCK DIAGRAM OF FLEXIBLE BOOSTER DYNAMICS
FIGURE 3-2

The performance is thus measured by the square of the drift at the terminal time plus a positive constant times the integral of the bending moment squared. This performance index chosen is essentially a penalty function approximation to minimizing the square of the terminal drift subject to an upper limit on the square of the average moment. The penalty function method has previously been effectively used to handle inequality constraints [10] . From a practical standpoint, use of (3-17) will result in a linear controller in which the gain programs can be computed with little difficulty.

Writing (3-17) in vector-matrix notation facilitates subsequent computations. In this notation (3-17) becomes

$$x_0(T) = \frac{1}{2}\left[x'(T)Rx(T) + \int_t^T (x'Fx + x'L'u + u'Lx + u'Gu)\, d\tau\right] \tag{3-18}$$

where

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \mu_\alpha k^2 \begin{bmatrix} \mu_\alpha/V^2 & 0 & 0 & -\mu_\alpha/V & -b/V & -c/V \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu_\alpha/V & 0 & 0 & \mu_\alpha & b & c \\ -b/V & 0 & 0 & b & (bk)^2 & bc/\mu_\alpha \\ -c/V & 0 & 0 & c & bc/\mu_\alpha & c^2 \end{bmatrix}$$

$$L = \mu_c k^2 \left[-\mu_\alpha/v\ ,\ 0\ ,\ 0\ ,\ \mu_\alpha\ ,\ b\ ,\ c\right]$$

and $G = \mu_c^2 k^2$

The exact Hamiltonian is

$$h = (p_0/2)\{x'[\mathcal{A}'R + R\mathcal{A} + F]x + x'(R\beta + L')u + u'(\beta'R + L)x + u'Gu\} + p'(\mathcal{A}x + \beta u) \tag{3-19}$$

The Maximum Principle yields

$$u = -G^{-1}[(\beta'R + L)x + \beta'p/p_0] \tag{3-20}$$

Utilizing (3-20) in (3-19) results in

$$h = (p_0/2)x'[(\mathcal{A}' - L'G^{-1}\beta')R + R(\mathcal{A} - \beta G^{-1}L) - R\beta G^{-1}\beta'R - L'G^{-1}L + F]x$$

$$- (1/2p_0)p'\beta G^{-1}\beta'p + p'[\mathcal{A} - \beta G^{-1}(\beta'R + L)]x$$

The corresponding canonical equations are given by

$$\begin{bmatrix} \dot{x} \\ \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \mathcal{A} - \beta G^{-1}(\beta'R + L) & \beta G^{-1}\beta' \\ \\ Q & -[\mathcal{A}' - (R\beta + L')G^{-1}\beta'] \end{bmatrix} \begin{bmatrix} x \\ \\ p \end{bmatrix} \tag{3-21}$$

where $\quad Q = (\mathcal{A}' - LG^{-1}\beta')R + R(\mathcal{A} - \beta G^{-1}L) - R\beta G^{-1}\beta'R - L'G^{-1}L + F$

Since $x(T)$ is free, the boundary conditions are

$$\begin{aligned} & p(T) = 0 \\ \text{except} \quad & p_0(T) = -1 \end{aligned} \tag{3-22}$$

The submatrices of (3-21) are given by

$$\mathcal{A} - \beta G^{-1}(\beta'R + L) = \left[\begin{array}{cccc:cc} -(d_1 + a_T\mu_\alpha/\mu_c V) & 0 & 0 & (h + a_T\mu_\alpha/\mu_c) & (a_T b/\mu_c - \ell) & (a_T c/\mu_c + n) \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline (-w + \mu_\alpha q/\mu_c)/V & 0 & 0 & (w - \mu_\alpha q/\mu_c) & (d_6 - bq/\mu_c) & (\omega^2 - cq/\mu_c) \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{3-23}$$

$$\beta G^{-1}\beta' = \left[\begin{array}{cccc:cc}(a_T/\mu_c k)^2 & 0 & -a_T/\mu_c k^2 & 0 & -a_T q/\mu_c^2 k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -a_T/\mu_c k^2 & 0 & 1/k^2 & 0 & q/\mu_c k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -a_T q/\mu_c^2 k^2 & 0 & q/\mu_c k^2 & 0 & (q/\mu_c k)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right] \quad (3\text{-}24)$$

and

$$Q = \left[\begin{array}{cccc:cc} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (3\text{-}25)$$

The above matrices have been partitioned so that the upper left-hand 4×4 matrix is that resulting from the rigid-body dynamics.

For subsequent use, it is desirable to write the $\dot{p}$ equations of (3-21) in component form:

$$\dot{p}_0 = 0 \quad (3\text{-}26a)$$

$$\dot{p}_1 = -p_0 x_2 + p_1\left(d_1 + \frac{a_T \mu_\alpha}{\mu_c V}\right) - p_2 + p_5\left(\frac{w}{V} - \frac{\mu_\alpha q}{\mu_c V}\right) \quad (3\text{-}26b)$$

$$\dot{p}_2 = -p_0 x_1 \quad (3\text{-}26c)$$

$$\dot{p}_3 = -p_4 \quad (3\text{-}26d)$$

$$\dot{p}_4 = -p_1\left(h + \frac{a_T \mu_\alpha}{\mu_c}\right) - p_5\left(w - \frac{\mu_\alpha q}{\mu_c}\right) \quad (3\text{-}26e)$$

$$\dot{p}_5 = p_1(\ell - a_T b/\mu_c) - p_5(d_6 - bq/\mu_c) - p_6 \quad (3\text{-}26f)$$

$$\dot{p}_6 = -p_1(n + a_T c/\mu_c) - p_5(\omega^2 - cq/\mu_c) \quad (3\text{-}26g)$$

Simplified Problem - Letting $\omega^2 \to \infty$ in (3-16) and further requiring that the product $\omega^2 x_6$ remain finite during the transition requires that x_6 and hence x_5 approach zero. Inspection of (3-26g) with the above reasoning results in the conclusion that p_5 tends to zero. Hence the simplified system is of fourth order. In accordance with the theory developed, the state of the complete process is represented by

$$x = X + \xi$$

where

$$X = \begin{bmatrix} v_{av} \\ y_{av} \\ \dot{\theta}_{av} \\ \theta_{av} \\ 0 \\ 0 \end{bmatrix} \qquad \xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dot{\varphi} \\ \varphi \end{bmatrix}$$

Thus (3-16) is reduced to the simplified equations of motion by taking only the first four elements of the exact state vector:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} -d_1 & 0 & 0 & h \\ 1 & 0 & 0 & 0 \\ -\mu_\alpha/V & 0 & 0 & \mu_\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} -a_T \\ 0 \\ \mu_c \\ 0 \end{bmatrix} \delta \tag{3-27}$$

The block diagram for the simplified system is obtained from Figure 3-2 by removing the portion shown by the light-weight lines.

The optimum control law (3-20) is written as

$$u = U + qp_5/(\mu_c k)^2 - (bx_5 + c\, x_6)/\mu_c \tag{3-28}$$

where U is the optimum control law for the simplified process, which, from (3-20), is given by

$$U = \frac{\mu_\alpha}{\mu_c} \left(\frac{X_1}{V} - X_4\right) + \frac{1}{\mu_c^2 k^2} (\mu_c P_3 - a_T P_1) \tag{3-29}$$

The exact canonical equations, given by (3-21), include those of the simplified problem. The simplified canonical equations are given by

$$\begin{bmatrix} \dot{X} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} X \\ P \end{bmatrix} \tag{3-30}$$

where H_{11}, H_{12} and H_{21} are given by the partitioned 4 x 4 elements of (3-23), (3-24) and (3-25) respectively and $H'_{22} = -H_{11}$. By assuming a time-invariant model, the coefficients of the H_{ij} in (3-30) are constant and the solution to (3-30) is obtained by use of the Laplace transform. In particular

$$\Phi(t) = \mathcal{L}^{-1}\left\{\begin{bmatrix} (sI - H_{11}) & -H_{12} \\ -H_{12} & (sI - H_{22}) \end{bmatrix}^{-1}\right\}$$

Hence the solution to (3-30) for $[X(T), P(T)]$ in terms of a starting state $[X(t), P(t)]$ can be written as follows:

$$\begin{aligned} X(T) &= \Phi_{11}(T - t)X(t) + \Phi_{12}(T - t)P(t) \\ P(T) &= \Phi_{21}(T - t)X(t) + \Phi_{22}(T - t)P(t) \end{aligned} \tag{3-31}$$

where

$$\Phi(T - t) = \begin{bmatrix} \Phi_{11}(T - t) & \Phi_{12}(T - t) \\ \Phi_{21}(T - t) & \Phi_{22}(T - t) \end{bmatrix}$$

is the "fundamental matrix [11] corresponding to (3-30). Inasmuch as there are no terminal constraints on the state vector, the costate vector is identically zero (3-22). Hence, it follows from (3-31) that

$$P(t) = M(\tau)X(t) \tag{3-32}$$

where $$M(\tau) = -\Phi_{22}^{-1}(\tau)\Phi_{21}(\tau) \tag{3-33}$$

and $\tau = T - t$ (time-to-go)

Performing the operations indicated in (3-33) results in

$$M = \frac{1}{\Delta}\begin{bmatrix} \frac{(1-e^{\theta})(1-e^{-\theta})}{\zeta^2} & \frac{1-e^{-\theta}}{\zeta} & \frac{\lambda(1-e^{\theta})f_1}{\zeta^2} & \frac{\lambda(1-e^{\theta})f_2}{\zeta} \\ \frac{1-e^{\theta}}{\zeta} & \Delta - e^{\theta} & -\frac{\lambda e^{\theta} f_1}{\zeta} & -\lambda e^{\theta} f_2 \\ \frac{\lambda(1-e^{\theta})f_1}{\zeta^2} & -\frac{\lambda e^{\theta} f_1}{\zeta} & -\frac{\lambda^2 e^{\theta} f_1^2}{\zeta^2} & -\frac{\lambda^2 e^{\theta} f_1 f_2}{\zeta} \\ \frac{\lambda(1-e^{\theta})f_2}{\zeta} & -\lambda e^{\theta} f_2 & -\frac{\lambda^2 e^{\theta} f_1 f_2}{\zeta} & -\lambda^2 e^{\theta} f_2^2 \end{bmatrix} \qquad (3\text{-}34)$$

where $\theta = \zeta\tau$ (dimensionless time-to-go)

$$\zeta = d_1 + \frac{\mu_\alpha a_T}{\mu_c V}$$

$$f_1 = \frac{\theta^2}{2} - \theta + 1 - e^{-\theta}$$

$$f_2 = \theta - 1 + e^{-\theta}$$

$$\lambda = \frac{1}{\zeta^2}\left[h + \frac{\mu_\alpha a_T}{\mu_c}\right]$$

and

$$\Delta\zeta^3 k^2 = e^{\theta}\left[\lambda^2\left(\frac{\theta^5}{20} - \frac{\theta^4}{4} + \frac{\theta^3}{3}\right) - \lambda(\eta - \lambda)\left(\frac{\theta^3}{3} - \theta^2\right) - (\eta - \lambda)^2\left(\theta - \frac{3}{2}\right) + \zeta^3 k^2\right]$$

$$+ \frac{e^{-\theta}}{2}(\eta - \lambda)^2 + (\eta - \lambda)[2(\eta - \lambda) - \lambda\theta^2]$$

where $\eta = \frac{a_T}{\mu_c} = \frac{J_{TOT}}{Mg}$

Substitution of (3-34) into (3-32) and the result into (3-29) gives the following expression for the control law for the rigid-body

$$U = \left[\frac{\mu_\alpha}{\mu_c V} + \Gamma \frac{(1 - e^{-\theta})}{\zeta}\right] X_1 + \Gamma X_2 + \Gamma \frac{\lambda f_1}{\zeta} X_3 + \left[-\frac{\mu_\alpha}{\mu_c} + \Gamma \lambda f_2\right] X_4 \qquad (3\text{-}35)$$

where $$\Gamma = \frac{1}{\Delta}\left[-\lambda e^{\theta} f_1 - a_T \frac{(1 - e^{\theta})}{\mu_c}\right]$$

A study was undertaken using the equations of motion (3-27) and the associated optimum control law (3-35) in order to determine acceptable values of the parameter K which weights the moment integral to the drift in the performance index. The vehicle parameters which were used in this investigation are given in Table 3-1. The result was that for K lying between one and two, both the terminal drift and the maximum inflight moment were acceptable. For K near zero, the terminal drift was least, but the maximum inflight moment was unacceptably large; while for K of five, the terminal drift diverged.

Quasi-Optimum Controller - The quasi-optimum control law is now constructed from (3-28). It is noted that, since the exact costate vector has six components and the simplified problem was of fourth order, the m_{5i} and m_{6i} terms of the gain matrix $M(\tau)$, now 6×6, must be found. Since p_5 was previously shown to be zero in the simplified state, $m_{5i} = 0$. Hence, from (3-26f) with $p_5 = 0$

$$P_6 = (\ell - a_T b/\mu_c) P_1 \qquad (3\text{-}36)$$

where P_1 is known from (3-32); m_{66} is still undetermined but is not required in the quasi-optimum control law since p_6 does not appear. The full gain matrix $M(\tau)$ can thus be written as

$$M(\tau) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & m_{11}(\ell - a_T b/\mu_c) \\ m_{12} & m_{22} & m_{23} & m_{24} & 0 & m_{12}(\ell - a_T b/\mu_c) \\ m_{13} & m_{23} & m_{33} & m_{34} & 0 & m_{13}(\ell - a_T b/\mu_c) \\ m_{14} & m_{24} & m_{34} & m_{44} & 0 & m_{14}(\ell - a_T b/\mu_c) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_{11}(\ell - a_T b/\mu_c) & m_{12}(\ell - a_T b/\mu_c) & m_{13}(\ell - a_T b/\mu_c) & m_{14}(\ell - a_T b/\mu_c) & 0 & m_{66} \end{bmatrix} \qquad (3\text{-}37)$$

where the m_{ij} ($i = 1, 2, 3, 4$) are given in (3-34).

Thus, substituting (3-37) into (3-28), the quasi-optimum control law is

$$u = U - \frac{b}{\mu_c} x_6 + [- \frac{c}{\mu_c} + \frac{\Gamma}{k^2 \mu_c \zeta^2} (\ell - \frac{a_T b}{\mu_c})(1 - e^{-\theta})] x_5 \tag{3-38}$$

where U is given in (3-35).

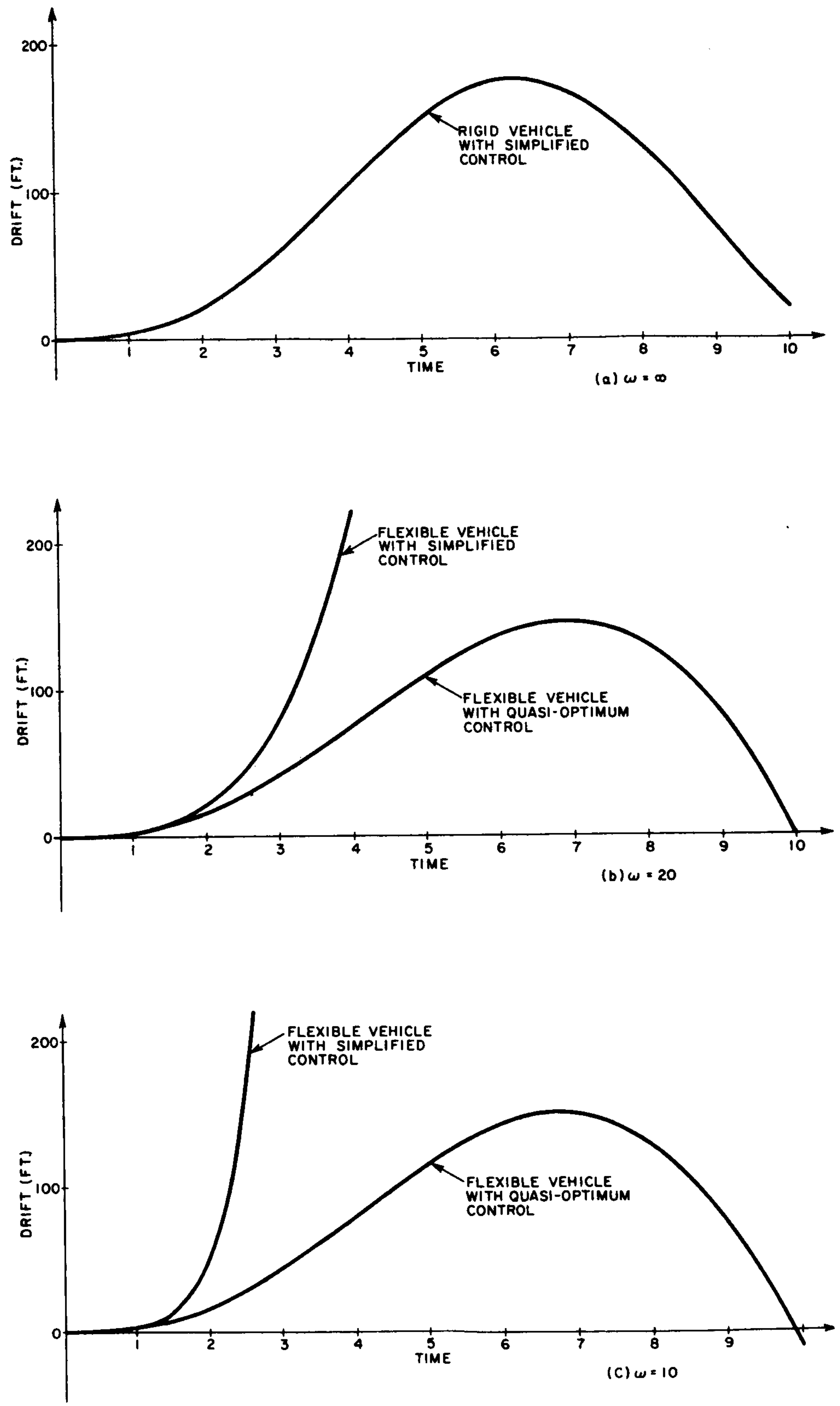
To verify the acceptability of (3-38), a simulation of the motion of the flexible vehicle (3-16) using both the simplified control law (3-35) and the quasi-optimum control law (3-38), with $K = 1$, was performed with the aid of an IBM 7040 digital computer.

The final results for $\omega = \infty$ (rigid body), $\omega = 20$ and $\omega = 10$ are shown in Figures 3-3 through 3-6. It is interesting to note the behavior of the control signal (Figure 3-5) with regard to the angle of flexure (Figure 3-6). The quasi-optimum control law seems to be tuned to variations in the angle of flexure, and almost 180 degrees out of phase with it. Note further that the simplified control law when used with the flexible vehicle does a better job as ω grows. This is to be expected inasmuch as it is exact at $\omega = \infty$. As the system becomes more and more flexible (i.e., as $\omega \to 0$), the performance using the quasi-optimum control law deteriorates, again as expected: the quasi-optimum control law is only a first order correction to that for a rigid vehicle. For $\omega = 5$, it was found that the system behavior was not satisfactory; all quantities ultimately diverged.

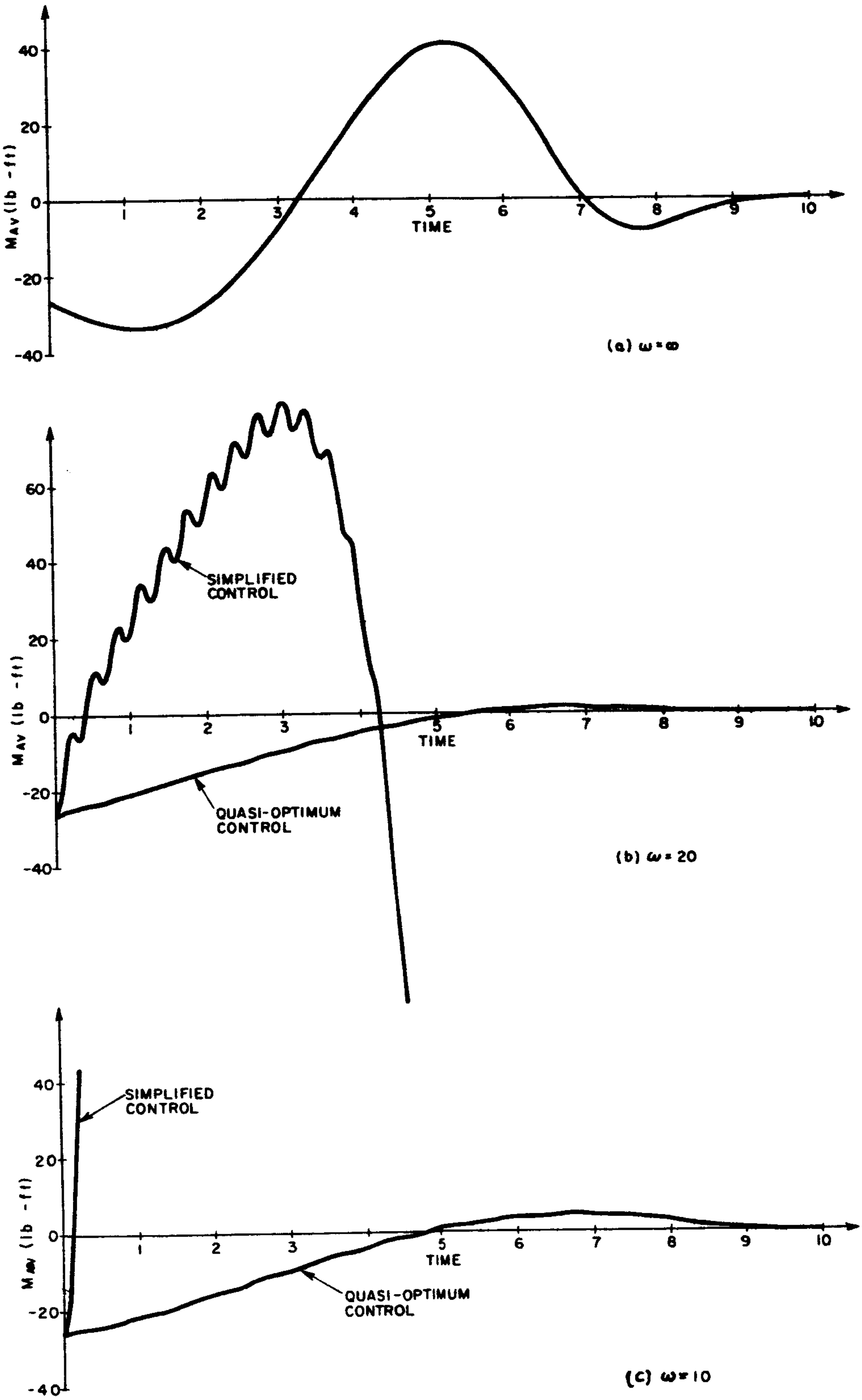
The vehicle considered here was aerodynamically unstable (center of pressure was forward of the center of gravity). It is quite possible that with a more aerodynamically stable vehicle ω could be made closer to zero and still permit use of the quasi-optimum control law.

TABLE 3-1 - VEHICLE PARAMETERS IN SIMULATION

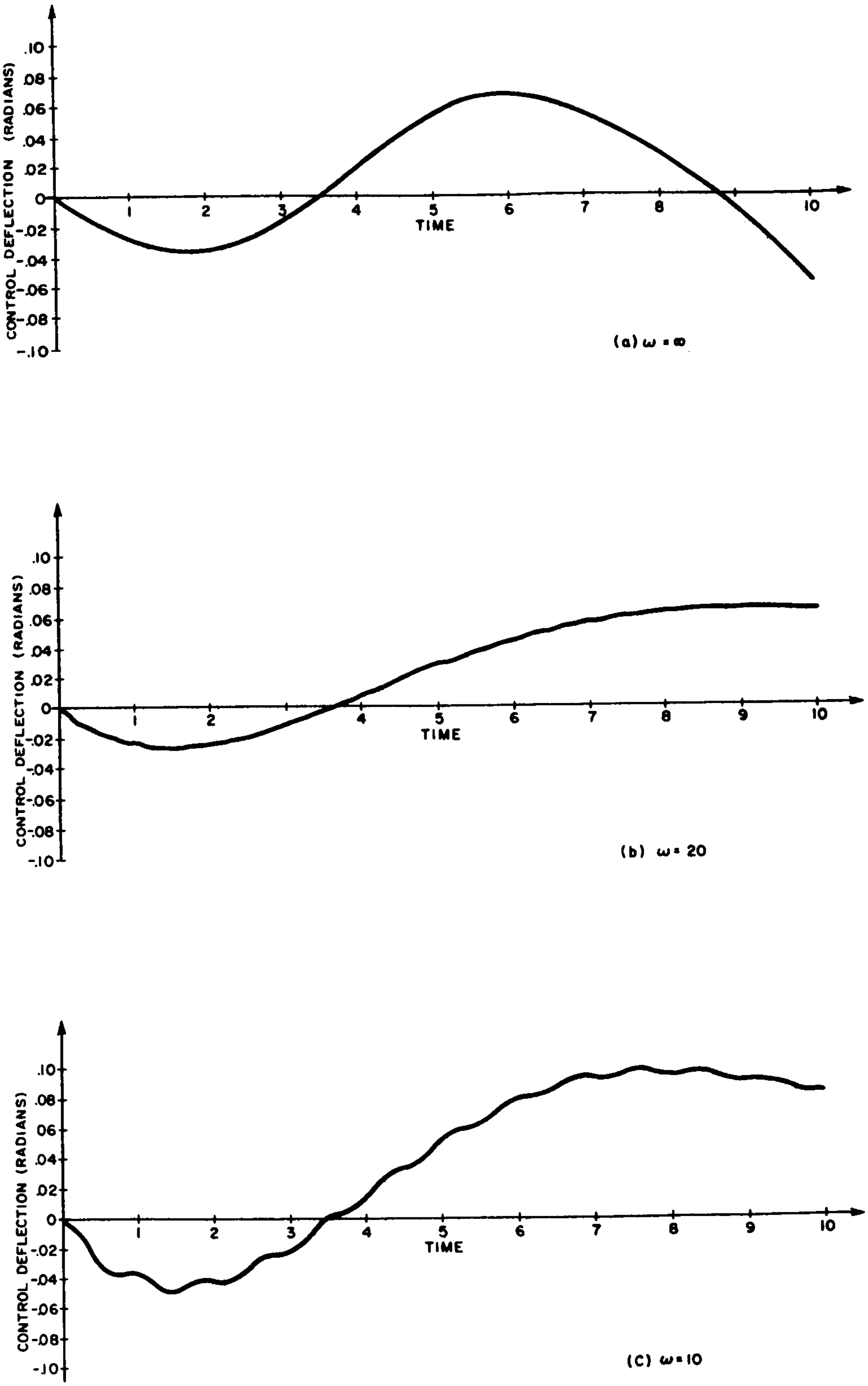
m_1	-	35 slugs
m_2	-	15 slugs
T	-	3800 lb
V	-	600 ft/sec
$C_{L\alpha 1}$	-	2.66 rad.^{-1}
$C_{L\alpha 2}$	-	2.66 rad.^{-1}
C_{D1}	-	.06
C_{D2}	-	.06
g_1	-	3.125 ft
ℓ_1	-	6.25 ft
g_2	-	1.67 ft
S_1	-	0.833 ft
S_2	-	0.667 ft
J_1	-	114 slug-ft^2
J_2	-	48.8 slug-ft^2
ρ_{air}	-	.00237 slugs/ft^3
S_1'	-	1.4 ft^2
S_2'	-	1.4 ft^2
T (terminal time)	-	10
C_α	-	$\begin{cases} 5470 \text{ lb-ft} & (\omega = 10) \\ 15790 \text{ lb-ft} & (\omega = 20) \end{cases}$



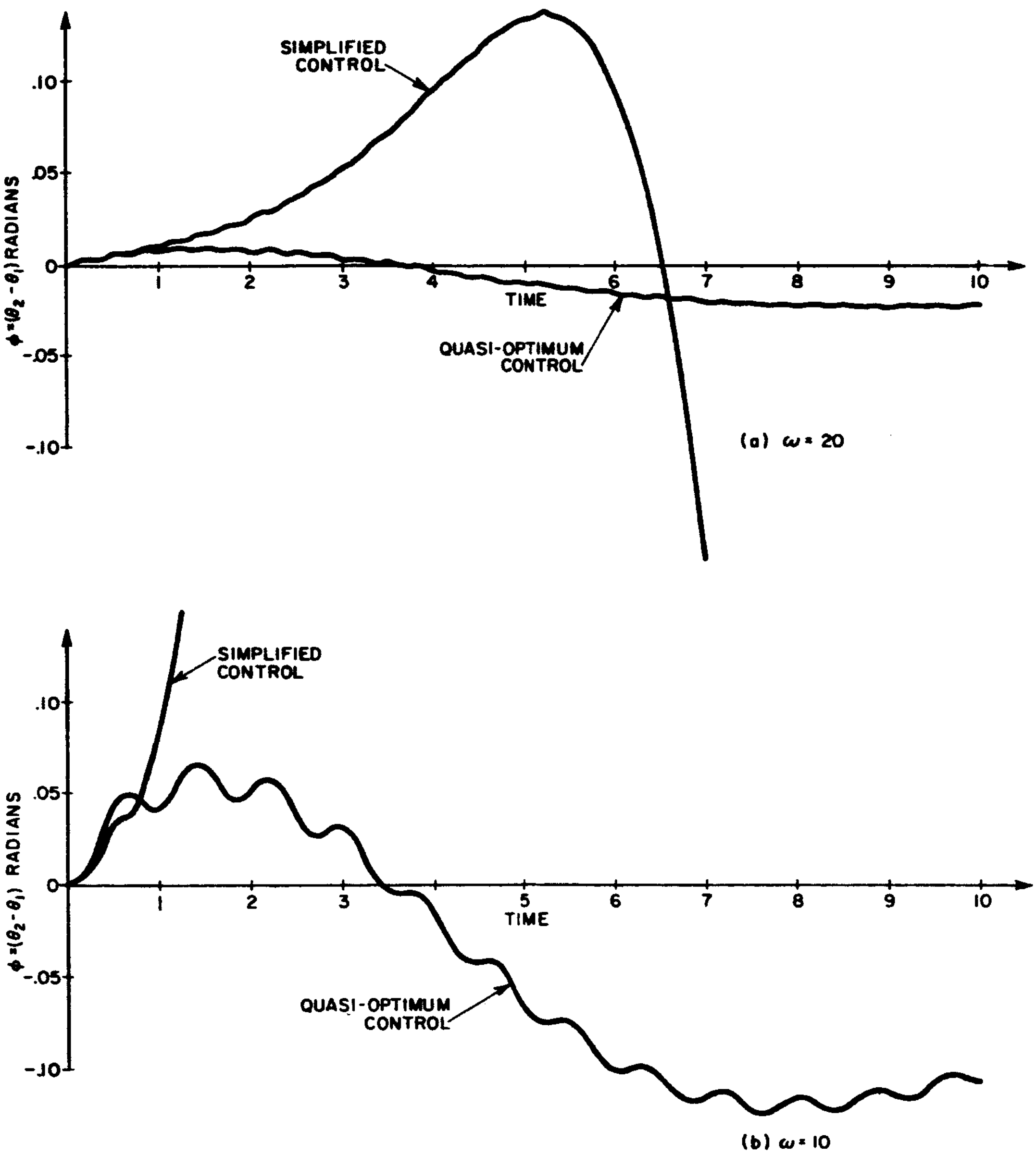
DRIFT NORMAL TO TRAJECTORY PLANE
FIGURE 3-3



AVERAGE AERODYNAMIC MOMENT
FIGURE 3-4



CONTROL DEFLECTION
FIGURE 3-5



ANGLE OF FLEXURE
FIGURE 3-6

2.4 TIME-OPTIMUM 3-AXIS ATTITUDE CONTROL OF A SPACE VEHICLE

Another problem which was begun was that of controlling the attitude of a space vehicle in which the gyroscopic coupling torques are small but not negligible.

Exact Problem - The equations governing the motion of the vehicle are taken as

$$\begin{aligned} d\theta_i/dt &= \omega_i \qquad\qquad i, j, k = 1, 2, 3 \\ d\omega_i/dt &= [(I_j - I_k)\omega_j\omega_k + c_i f_i]/I_i \end{aligned} \tag{4-1}$$

where

$\theta_i = x_i$ = angular position $\qquad i = 1, 2, 3$

$\omega_i = x_{3+i}$ = angular velocity $\qquad i = 1, 2, 3$

I_i = moment of inertial about the principal axis

c_i = moment arm of jet control

f_i = thrust of jet control

The thrusts are assumed to be bounded in amplitude:

$$|f_i(t)| \le M_i \qquad i = 1, 2, 3 \tag{4-2}$$

The cross-axis inertia ratios $(I_j - I_k)/I_i$ are assumed to be small but nonzero, and are represented by additional state variables

$$x_{6+i} = (I_j - I_k)/I_i \qquad i = 1, 2, 3 \tag{4-3}$$

Introduction of time as the performance index which is to be minimized produces an additional state variable $x_0(t) = t$. Hence the state equations can be written as

$$
\begin{aligned}
\dot{x}_0 &= 1 \\
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= x_7x_5x_6 + k_1u_1(t) \\
\dot{x}_5 &= x_8x_6x_4 + k_2u_2(t) \\
\dot{x}_6 &= x_9x_4x_5 + k_3u_3(t) \\
\dot{x}_7 &= \dot{x}_8 = \dot{x}_9 = 0
\end{aligned}
\qquad (4\text{-}4)
$$

where

$$
\begin{aligned}
k_i &= c_i M_i / I_i \\
|u_i(t)| &\leq 1
\end{aligned}
\qquad i = 1, 2, 3 \qquad (4\text{-}5)
$$

The Hamiltonian for the exact system is

$$
\begin{aligned}
h = {} & p_0 + p_1x_4 + p_2x_5 + p_3x_6 + p_4x_7x_5x_6 + p_4u_1k_1 \\
& + p_5x_8x_6x_4 + p_5u_2k_2 + p_6x_9x_4x_5 + p_6u_3k_3
\end{aligned}
\qquad (4\text{-}6)
$$

where

$$
\dot{p}_0 = \dot{p}_1 = \dot{p}_2 = \dot{p}_3 = 0 \qquad (4\text{-}7)
$$

and

$$
\begin{aligned}
\dot{p}_4 &= -p_1 - p_5x_8x_6 - p_6x_9x_5 \\
\dot{p}_5 &= -p_2 - p_4x_7x_6 - p_6x_9x_4 \\
\dot{p}_6 &= -p_3 - p_4x_7x_5 - p_5x_8x_4 \\
\dot{p}_7 &= -p_4x_5x_6 \\
\dot{p}_8 &= -p_5x_6x_4 \\
\dot{p}_9 &= -p_6x_4x_5
\end{aligned}
$$

The Maximum Principle yields the following control law

$$u_1 = \text{sgn}\,(k_1 p_4)$$

$$u_2 = \text{sgn}\,(k_2 p_5) \qquad (4\text{-}8)$$

$$u_3 = \text{sgn}\,(k_3 p_6)$$

Simplified Problem - The simplified system is defined by

$$x_7 = x_8 = x_9 = 0 \qquad (4\text{-}9)$$

Thus, for the simplified system the three axes are uncoupled, and from the solution to the well-known Bushaw problem, we have the solution to the simplified problem

$$U_1 = \text{sgn}\; k_1(-P_{10}(\tau - t) + P_{40})$$

$$U_2 = \text{sgn}\; k_2(-P_{20}(\tau - t) + P_{50}) \qquad (4\text{-}10)$$

$$U_3 = \text{sgn}\; k_3(-P_{30}(\tau - t) + P_{60})$$

where P_i , $i = 1, \ldots, 6$, are the adjoint variables for the simplified system.

Quasi-Optimum Control Law - The quasi-optimum control law is given by

$$u_1 = \text{sgn}\,[k_1(P_4 + m_{47}x_7 + m_{48}x_8 + m_{49}x_9)]$$

$$u_2 = \text{sgn}\,[k_2(P_5 + m_{57}x_7 + m_{58}x_8 + m_{59}x_9)] \qquad (4.11)$$

$$u_3 = \text{sgn}\,[k_3(P_6 + m_{67}x_7 + m_{68}x_8 + m_{69}x_9)]$$

where P_4, P_5, P_6 are adjoint variables of the simplified system defined above, and the m_{ij} are components of the gain matrix M which is obtained by finding the fundamental matrix for the auxiliary system (20) the coefficient matrices of which are given by

$$H_{XP} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_5X_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_6X_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_4X_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-12}$$

$$H_{PX} = H'_{XP} \tag{4-13}$$

$$H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_5X_6 & P_6X_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_4X_6 & 0 & P_6X_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_4X_5 & P_5X_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_4X_6 & P_4X_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_5X_6 & 0 & P_5X_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_6X_5 & P_6X_4 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-14}$$

$$
H_{PP} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2k_1\delta(k_1P_4) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2k_2\delta(k_2P_5) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2k_3\delta(k_3P_6) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{4-15}
$$

In component form the auxiliary equations are

$$
\begin{aligned}
\dot{\xi}_0 &= 0 & \dot{\psi}_0 &= 0 \\
\dot{\xi}_1 &= \xi_4 & \dot{\psi}_1 &= 0 \\
\dot{\xi}_2 &= \xi_5 & \dot{\psi}_2 &= 0 \\
\dot{\xi}_3 &= \xi_6 & \dot{\psi}_3 &= 0 \\
\dot{\xi}_4 &= X_5X_6\xi_7 + 2k_1\delta(k_1P_4)\psi_4 & \dot{\psi}_4 &= -\psi_1 - P_5X_6\xi_8 - P_6X_5\xi_9 \\
\dot{\xi}_5 &= X_6X_4\xi_8 + 2k_2\delta(k_2P_5)\psi_5 & \dot{\psi}_5 &= -\psi_2 - P_4X_6\xi_7 - P_6X_4\xi_9 \\
\dot{\xi}_6 &= X_4X_5\xi_9 + 2k_3\delta(k_3P_6)\psi_6 & \dot{\psi}_6 &= -\psi_3 - P_4X_5\xi_7 - P_5X_4\xi_8 \\
\dot{\xi}_7 &= 0 & \dot{\psi}_7 &= -X_5X_6\psi_4 - P_4X_6\xi_5 - P_4X_5\xi_6 \\
\dot{\xi}_8 &= 0 & \dot{\psi}_8 &= -X_6X_4\psi_5 - P_5X_6\xi_4 - P_5X_4\xi_6 \\
\dot{\xi}_9 &= 0 & \dot{\psi}_9 &= -X_4X_5\psi_6 - P_6X_5\xi_4 - P_6X_4\xi_5
\end{aligned} \tag{4-16}
$$

These equations can be integrated by using the same procedure as used in the quasi-optimum solution to the Bushaw problem. Assume that the solution to the simplified problem is

$$
U_i = \begin{cases} U_i & t < t_{si} \\ -U_i & t > t_{si} \end{cases} \qquad i = 1, 2, 3 \tag{4-17}
$$

and that $t_{s1} < t_{s2} < t_{s3}$. Then the fundamental matrix $\Phi(T, 0)$ for (4-16) is

$$\Phi(T, 0) = \Phi(T, t_{s3}^+)\Phi(t_{s3}^+, t_{s3}^-)\Phi(t_{s3}^-, t_{s2}^+)\Phi(t_{s2}^+, t_{s2}^-)\Phi(t_{s2}^-, t_{s1}^+)\Phi(t_{s1}^+, t_{s1}^-)\Phi(t_{s1}^-, 0) \tag{4-18}$$

where

$$0 < t_{s1} < t_{s2} < t_{s3} < T .$$

The matrices $\Phi(t_{si}^+, t_{si}^-)$ contain terms resulting from the impulses in H_{pp} . Using the method of the first example, it is found that

$$\begin{aligned} &\Phi_{11}(t_{si}^+, t_{si}^-) = \Phi_{22}(t_{si}^+, t_{si}^-) = I \\ &\Phi_{21}(t_{si}^+, t_{si}^-) = 0, \qquad i = 1, 2, 3 \end{aligned} \tag{4-19}$$

and that the elements of $\Phi_{12}(t_{si}^+, t_{si}^-)$ are all zero except for the element in the $(4 + i, 4 + i)$ position which is $2U_i/P_i$. Furthermore, it can be shown that, if $\tau = t - t_{si}^+$, $i = 0, 1, 2, 3$, $t_{si}^+ \le t \le t_{s(i+1)}^-$, $t_{s4} = T$, $t_{s0} = 0$, then

$$\Phi(t, t_{si}^+) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{4-20}$$

where

$$\Phi_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Phi_{14} & 0 & 0 & \Phi_{17} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \Phi_{25} & 0 & 0 & \Phi_{28} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \Phi_{36} & 0 & 0 & \Phi_{39} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \Phi_{47} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \Phi_{58} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \Phi_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4-21}$$

$$\Phi_{12} = 0 \tag{4-22}$$

$$\Phi_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi_{48} & \Phi_{49} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi_{57} & 0 & \Phi_{59} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi_{67} & \Phi_{68} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{75} & \Phi_{76} & 0 & \Phi_{78} & \Phi_{79} \\ 0 & 0 & 0 & 0 & \Phi_{84} & 0 & \Phi_{86} & \Phi_{87} & 0 & \Phi_{89} \\ 0 & 0 & 0 & 0 & \Phi_{94} & \Phi_{95} & 0 & \Phi_{97} & \Phi_{98} & 0 \end{bmatrix} \tag{4-23}$$

$$\Phi_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi_{41} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi_{52} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi_{63} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \Phi_{71} & 0 & 0 & \Phi_{74} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \Phi_{82} & 0 & 0 & \Phi_{85} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Phi_{93} & 0 & 0 & \Phi_{96} & 0 & 0 & 1 \end{bmatrix} \tag{4-24}$$

where, if X_l and P_l denote the value of the state and adjoint variables at the beginning of the interval and U_l denotes the value of the control (either positive or negative according to (4-17)), then

$$\Phi_{14} = \tau$$

$$\Phi_{17} = \frac{1}{12} k_2 k_3 U_2 U_3 \tau^4 + \frac{1}{6}(X_6 k_2 U_2 + X_5 k_3 U_3)\tau^3 + \frac{1}{2} X_5 X_6 \tau^2$$

$$\Phi_{25} = \tau$$

$$\Phi_{28} = \frac{1}{12} k_1 k_3 U_1 U_3 \tau^4 + \frac{1}{6}(X_6 k_1 U_1 + X_4 k_3 U_3)\tau^3 + \frac{1}{2} X_4 X_6 \tau^2$$

$$\Phi_{36} = \tau$$

$$\Phi_{39} = \frac{1}{12}k_1k_2U_1U_2\tau^4 + \frac{1}{6}(X_4k_2U_2 + X_5k_1U_1)\tau^3 + \frac{1}{2}\gamma_4X_5\tau^2$$

$$\Phi_{47} = \frac{1}{3}k_2k_3U_2U_3\tau^3 + \frac{1}{2}(X_6k_2U_2 + X_5k_3U_3)\tau^2 + X_5X_6\tau$$

$$\Phi_{58} = \frac{1}{3}k_1k_3U_1U_3\tau^3 + \frac{1}{2}(X_6k_1U_1 + X_4k_3U_3)\tau^2 + X_4X_6\tau$$

$$\Phi_{69} = \frac{1}{3}k_1k_2U_1U_2\tau^3 + \frac{1}{2}(X_4k_2U_2 + X_5k_1U_1)\tau^2 + X_4X_5\tau$$

$$\Phi_{48} = \frac{1}{3}k_3U_3P_2\tau^3 - \frac{1}{2}(-X_6P_2 + k_3U_3P_5)\tau^2 - P_5X_6\tau$$

$$\Phi_{49} = \frac{1}{3}k_2U_2P_3\tau^3 - \frac{1}{2}(-X_5P_3 + k_2U_2P_6)\tau^2 - P_6X_5\tau$$

$$\Phi_{41} = -\tau$$

$$\Phi_{57} = \frac{1}{3}k_3U_3P_1\tau^3 - \frac{1}{2}(-X_6P_1 + k_3U_3P_4)\tau^2 - P_4X_6\tau$$

$$\Phi_{59} = \frac{1}{3}k_1U_1P_3\tau^3 - \frac{1}{2}(-X_4P_3 + k_1U_1P_6)\tau^2 - P_6X_4\tau$$

$$\Phi_{52} = -\tau$$

$$\Phi_{67} = \frac{1}{3}k_2U_2P_1\tau^3 - \frac{1}{2}(-X_5P_1 + k_2U_2P_4)\tau^2 - P_4X_5\tau$$

$$\Phi_{68} = \frac{1}{3}k_1U_1P_2\tau^3 - \frac{1}{2}(-X_4P_2 + k_1U_1P_5)\tau^2 - P_5X_4\tau$$

$$\Phi_{63} = -\tau$$

$$\Phi_{75} = \frac{1}{3}k_3U_3P_1\tau^3 - \frac{1}{2}(k_3U_3P_4 - P_1X_6)\tau^2 - P_4X_6\tau$$

$$\Phi_{76} = \frac{1}{3}k_2U_2P_1\tau^3 - \frac{1}{2}(k_2U_2P_4 - P_1X_5)\tau^2 - P_4X_5\tau$$

$$
\begin{aligned}
\Phi_{78} = {} & \frac{1}{3}k_3^2[P_1k_1U_1 - P_2k_2U_2]\,\frac{1}{6}\tau^6 \\
& + [k_3^2(\frac{1}{2}X_4P_1 - \frac{1}{3}X_5P_2) - \frac{1}{3}k_3^2k_1U_1P_4 + \frac{1}{2}k_3^2k_2U_2P_5 - \frac{5}{6}k_2U_2k_3U_3X_6P_2 + \frac{5}{6}k_1U_1k_3U_3P_1X_6]\,\frac{1}{5}\tau^5 \\
& + [\frac{1}{2}k_3^2(X_5P_5 - P_4X_4) + (k_3U_3)[\frac{1}{2}X_4P_4X_6 - \frac{5}{6}X_5X_6P_2 + P_1X_4X_6] \\
& - \frac{1}{2}k_2U_2P_2X_6^2 + \frac{1}{2}k_1U_1P_4X_6^2 - \frac{5}{6}k_1k_3U_1U_3P_4X_6 + \frac{3}{2}k_2k_3U_2U_3P_5X_6]\,\frac{1}{4}\tau^4 \\
& + [-\frac{1}{2}k_1U_1P_4X_6^2 + \frac{3}{2}k_3U_3X_6(-P_4X_4 + P_5X_5) + k_2U_2X_6^2P_5 + X_6^2(P_1X_4 - \frac{1}{2}X_5P_2)]\,\frac{1}{3}\tau^3 \\
& + [P_5X_5 - P_4X_4]\frac{X_6^2}{2}\tau^2
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{79} = {} & \frac{1}{3}k_2^2(k_1U_1P_1 - k_3U_3P_3)\,\frac{1}{6}\tau^6 \\
& + [k_2^2(\frac{1}{2}P_1X_4 - \frac{1}{3}P_3X_6) + \frac{5}{6}k_1U_1k_2U_2P_1X_5 + \frac{1}{2}k_3U_3k_2^2P_6 - \frac{1}{3}k_1U_1k_2^2P_4 - \frac{5}{6}k_2k_3U_2U_3X_5P_3]\,\frac{1}{5}\tau^5 \\
& + [\frac{1}{2}k_2^2(X_6P_6 - X_4P_4) + k_2U_2(\frac{3}{2}P_1X_4X_5 - \frac{5}{6}X_5X_6P_3) - \frac{5}{6}k_1k_2U_1U_2P_4X_5 \\
& + \frac{3}{2}k_2k_3U_2U_3X_5P_6 + \frac{1}{2}k_1U_1P_1X_5^2 - \frac{1}{2}k_3U_3X_5^2]\,\frac{1}{4}\tau^4 \\
& + [\frac{3}{2}k_2U_2(P_6X_6X_5 - P_4X_4X_5) + k_3U_3P_6X_5^2 - \frac{1}{2}k_1U_1P_4X_5^2 - \frac{1}{2}X_5^2X_6P_3 + X_4X_5^2P_1]\,\frac{1}{3}\tau^3 \\
& + [-P_4X_4 - X_6P_6]X_5^2\frac{\tau^2}{2}
\end{aligned}
$$

$$\Phi_{71} = \frac{1}{4}k_2k_3U_2U_3\tau^4 + \frac{1}{3}(X_6k_2U_2 + X_5k_3U_3)\tau^3 + \frac{1}{2}X_5X_6\tau^2$$

$$\Phi_{74} = -\frac{1}{3}k_2k_3U_2U_3\tau^3 - \frac{1}{2}(X_6k_2U_2 + X_5k_3U_3)\tau^2 - X_5X_6\tau$$

$$\Phi_{84} = \frac{1}{3}k_3U_3P_2\tau^3 - \frac{1}{2}(k_3U_3P_5 - P_2X_6)\tau^2 - P_5X_6\tau$$

$$\Phi_{86} = \frac{1}{3}k_1U_1P_2\tau^3 - \frac{1}{2}(k_1U_1P_5 - P_2X_4)\tau^2 - P_5X_4\tau$$

$$\Phi_{87} = \frac{1}{3}k_3^2(k_2U_2P_2 - k_1U_1P_1)\frac{\tau^6}{6}$$

$$+ [-\frac{5}{6}k_1k_3U_1U_3P_1X_6 + \frac{5}{6}k_2k_3U_2U_3P_2X_6 + k_3^2(\frac{1}{2}P_2X_5 - \frac{1}{3}P_1X_4) - \frac{1}{3}k_2U_2k_3^2P_5 + \frac{1}{2}k_1U_1k_3^2P_4]\frac{\tau^5}{5}$$

$$+ [k_3U_3(\frac{3}{2}X_5X_6P_2 - \frac{5}{6}P_1X_4X_6 + \frac{1}{2}k_2U_2P_2X_6^2 - \frac{1}{2}k_1U_1P_1X_6^2$$

$$+ \frac{3}{2}k_1k_3U_1U_3P_4X_6 - \frac{5}{6}k_2k_3U_2U_3P_5X_6 + \frac{1}{2}k_3^2(P_4X_4 - P_5X_5)]\frac{\tau^4}{4}$$

$$+ [k_3U_3(\frac{3}{2}P_4X_4X_6 - \frac{3}{2}X_5P_5X_6) - \frac{1}{2}k_2U_2X_6^2P_5 + k_1U_1P_4X_6^2 + P_2X_5X_6^2 - \frac{1}{2}X_6^2P_1X_4]\ \frac{1}{3}\tau^3$$

$$+ [P_4X_4 - P_5X_5]X_6^2\frac{\tau^2}{2}$$

$$\Phi_{89} = \frac{1}{3}k_1^2(k_2U_2P_2 - k_3U_3P_3)\frac{\tau^6}{6}$$

$$+ [k_1^2(\frac{1}{2}P_2X_5 - \frac{1}{3}P_3X_6) - \frac{5}{6}k_1k_3U_1U_3P_3X_4 + \frac{5}{6}k_1k_2U_1U_2P_2X_4 - \frac{1}{3}k_1^2k_2U_2P_5 + \frac{1}{2}k_1^2k_3U_3P_6]\ \frac{1}{5}\tau^5$$

$$+ [k_1U_1(\frac{3}{2}P_2X_4X_5 - \frac{5}{6}P_3X_4X_6) + \frac{3}{2}k_1k_3U_1U_3P_6X_4 - \frac{5}{6}k_1k_2U_1U_2P_5X_4$$

$$+ \frac{1}{2}k_1^2(P_6X_6 - P_5X_5) - \frac{1}{2}k_3U_3P_3X_4^2 + \frac{1}{2}k_2U_2P_2X_4^2]\ \frac{1}{4}\tau^4$$

$$+ [\frac{3}{2}k_1U_1(X_4X_6P_6 - X_4X_5P_5) - \frac{1}{2}k_2U_2X_4^2P_5 + k_3U_3X_4^2P_6 + P_2X_4^2X_5 - \frac{1}{2}X_4^2X_6P_3]\ \frac{\tau^3}{3}$$

$$+ [P_6X_6 - P_5X_5]X_4^2\frac{\tau^2}{2}$$

$$\Phi_{82} = \frac{1}{4}k_1k_3U_1U_3\tau^4 + \frac{1}{3}(X_6k_1U_1 + X_4k_3U_3)\tau^3 + \frac{1}{2}X_4X_6\tau^2$$

$$\Phi_{85} = -\frac{1}{3}k_1k_3U_1U_3\tau^3 - \frac{1}{2}(X_6k_1U_1 + X_4k_3U_3)\tau^2 - X_4X_6\tau$$

$$\Phi_{94} = \frac{1}{3}k_2U_2P_3\tau^3 - \frac{1}{2}(k_2U_2P_6 - P_3X_5)\tau^2 - P_6X_5\tau$$

$$\Phi_{95} = \frac{1}{3}k_1U_1P_3\tau^3 - \frac{1}{2}(k_1U_1P_6 - P_3X_4)\tau^2 - P_6X_4\tau$$

$$\Phi_{97} = \frac{1}{3} k_2^2 (k_3 U_3 P_3 - k_1 U_1 P_1) \frac{1}{6} \tau^6$$

$$+ [- \frac{5}{6} k_1 k_2 U_1 U_2 P_1 X_5 + k_2^2 (\frac{1}{2} X_6 P_3 - \frac{1}{3} P_1 X_4) + \frac{5}{6} k_2 k_3 U_2 U_3 X_5 P_3 - \frac{1}{3} k_2^2 k_3 U_3 P_6 + \frac{1}{2} k_2^2 k_1 U_1 P_4] \frac{1}{5} \tau^5$$

$$+ [k_2 U_2 (\frac{3}{2} X_5 X_6 P_3 - \frac{5}{6} P_1 X_4 X_5) - \frac{5}{6} k_2 k_3 U_2 U_3 P_6 X_5 + \frac{3}{2} k_1 k_2 U_1 U_2 P_4 X_5$$

$$- \frac{1}{2} k_1 U_1 X_5^2 P_1 + \frac{1}{2} k_3 U_3 X_5^2 P_3 + \frac{1}{2} k_2^2 (X_4 P_4 - X_6 P_6)] \frac{\tau^4}{4}$$

$$+ [\frac{3}{2} k_2 U_2 (P_4 X_4 X_5 - X_5 X_6 P_6) - \frac{1}{2} k_3 U_3 X_5^2 P_6 - \frac{1}{2} X_5^2 X_4 P_1 + k_1 U_1 X_5^2 P_4 - \frac{1}{2} k_3 U_3 X_5^2 P_6] \frac{1}{3} \tau^3$$

$$+ [P_4 X_4 - P_6 X_6] X_5^2 \frac{\tau^2}{2}$$

$$\Phi_{98} = \frac{1}{3} k_1^2 (k_3 U_3 P_3 - k_2 U_2 P_2) \frac{1}{6} \tau^6$$

$$+ [- \frac{5}{6} k_1 k_2 U_1 U_2 P_2 X_4 + k_1^2 (\frac{1}{2} X_6 P_3 - \frac{1}{3} P_2 X_5) + \frac{5}{6} k_1 k_3 U_1 U_3 P_3 X_4 - \frac{1}{2} k_1^2 k_2 U_2 P_5 + \frac{1}{3} k_1^2 k_3 U_3 P_6] \frac{1}{5} \tau^5$$

$$+ [k_1 U_1 (- \frac{5}{6} X_4 P_2 X_5 + \frac{3}{2} P_3 X_4 X_6) - \frac{5}{6} k_1 k_3 U_1 U_3 P_6 X_4 - \frac{1}{2} k_2 U_2 P_2 X_4^2$$

$$+ \frac{3}{2} k_1 k_2 U_1 U_2 P_5 X_4 - \frac{1}{2} k_3 U_3 P_3 X_4^2 + \frac{k_1^2}{2} (P_5 X_5 - X_6 P_6)] \frac{1}{4} \tau^4$$

$$+ [\frac{3}{2} k_1 U_1 (X_4 X_5 P_5 - X_6 P_6 X_4) - \frac{1}{2} k_3 U_3 P_6 X_4^2 + k_2 U_2 P_5 X_4^2 - \frac{1}{2} X_4^2 X_5 P_2 - X_4^2 X_6 P_3] \frac{\tau^3}{3}$$

$$+ [P_5 X_5 - P_6 X_6] X_4^2 \frac{\tau^2}{2}$$

$$\Phi_{93} = \frac{1}{4} k_1 k_2 U_1 U_2 \tau^4 + \frac{1}{3} (k_1 U_1 X_5 + k_2 U_2 X_4) \tau^3 + \frac{1}{2} X_4 X_5 \tau^2$$

$$\Phi_{96} = - \frac{1}{3} k_1 k_2 U_1 U_2 \tau^3 - \frac{1}{2} (X_5 k_1 U_1 + X_4 k_2 U_2) \tau^2 - X_4 X_5 \tau$$

• The matrix $\Phi_{12}(T, 0)$, the (12) - component of $\Phi(T, 0)$ given in (4-18) has been calculated. Its elements $\hat{\Phi}_{ij}$ are all zero except

$$\hat{\Phi}_{22} = -2 \frac{U_1}{P_1} t_{s1}(t_{s2} + t_{s3} + T)$$

$$\hat{\Phi}_{25} = 2 \frac{U_1}{P_1} (t_{s2} + t_{s3} T)$$

$$\hat{\Phi}_{33} = -2 \frac{U_2}{P_2} (T + t_{s3})(t_{s1} + t_{s2})$$

$$\hat{\Phi}_{36} = 2 \frac{U_2}{P_2} (t_{s3} + T)$$

$$\hat{\Phi}_{44} = -2 \frac{U_3}{P_3} T(t_{s1} + t_{s2} + t_{s3})$$

$$\hat{\Phi}_{47} = 2T \frac{U_3}{P_3}$$

$$\hat{\Phi}_{52} = -2t_{s1} \frac{U_1}{P_1}$$

$$\hat{\Phi}_{55} = 2 \frac{U_1}{P_1}$$

$$\hat{\Phi}_{63} = -2 \frac{U_3}{P_3} (t_{s1} + t_{s2} + t_{s3})$$

$$\hat{\Phi}_{66} = 2 \frac{U_3}{P_3}$$

The other components of $\Phi(T, 0)$ can be found similarly.

2.5 MINIMUM LATERAL MISS-DISTANCE REENTRY

The problem of minimizing the lateral miss-distance at impact of a maneuverable re-entry vehicle is another problem which was considered. The problem was formulated in two dimensions over a flat earth in a uniform gravitational field.

Exact Problem - The relevent equations of motion are

$$\begin{aligned}
\dot{v}_x &= -C_d S\rho V v_x - C_\ell S\rho V v_y \\
\dot{v}_y &= C_\ell S\rho V v_x - C_d S\rho V v_y - g \\
\dot{x} &= v_x \\
\dot{y} &= v_y
\end{aligned} \tag{5-1}$$

where $V = (v_x^2 + v_y^2)^{\frac{1}{2}}$

x, y = coordinates of the vehicle, y being the altitude

v_x, v_y = components of velocity

C_ℓ = lift coefficient

C_d = drag coefficient

S = $\frac{1}{2}$ (reference area/mass)

ρ = density of the atmosphere

g = gravitational acceleration

The control variable was chosen as the lift coefficient C_ℓ which was assumed to be bounded in magnitude, hence

$$u = C_\ell ; \qquad |C_\ell| \leq \eta \tag{5-2}$$

The drag coefficient C_d was assumed to vary as $C_d = c + kC_\ell^2$, where c and k are constants. The density was assumed to vary exponentially

$$\rho = \rho_0 e^{-\beta y} \tag{5-3}$$

where ρ_0 and β are constants. Hence, the problem reduces to finding a feedback control u such that $x^2(T)$ is a minimum when $y(T) = 0$, where T is the terminal time. The variables $v_x(T)$, $v_y(T)$, $x(T)$ and T are treated as free variables.

By defining the state variables

$$x_0 = \tfrac{1}{2}x^2, \quad x_1 = x, \quad x_2 = y, \quad x_3 = v_x, \quad x_4 = v_y, \quad x_5 = S\rho V, \quad x_6 = k \tag{5-4}$$

the equations of motion can be rewritten as

$$\begin{aligned}
\dot{x}_0 &= x_1 x_3 \\
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -(c + x_6 u^2)\, x_3 x_5 - u x_4 x_5 \\
\dot{x}_4 &= u x_3 x_5 - (c + x_6 u^2)\, x_4 x_5 - g \\
\dot{x}_5 &= -(c + x_6 u^2) x_5^2 - x_4 x_5 \left(\frac{g}{V^2} + \beta\right) \\
\dot{x}_6 &= 0
\end{aligned} \tag{5-5}$$

Forming the exact Hamiltonian we have

$$\begin{aligned}
h = {} & p_0 x_1 x_3 + p_1 x_3 + p_2 x_4 + p_3 x_5 \left[-(c + x_6 u^2) x_3 - u x_4\right] + p_4 x_5 \left[u x_3 - (c + x_6 u^2) x_4 - \frac{g}{x_5}\right] \\
& - p_5 x_5 \left[(c + x_6 u^2) x_5 + \left(\frac{g}{V^2} + \beta\right) x_4\right]
\end{aligned} \tag{5-6}$$

Applying the Maximum Principle yields either

$$u = -\frac{1}{2x_6}\left[\frac{p_3 x_4 - p_4 x_3}{p_3 x_3 + p_4 x_4 + p_5 x_5}\right] \tag{5-7}$$

when

$$\frac{\partial^2 H}{\partial u^2} > 0 \quad \text{and} \quad \left|\frac{p_4 x_3 - p_3 x_4}{2x_6 (p_3 x_3 + p_4 x_4 + p_5 x_5)}\right| < \eta \tag{5-8}$$

or

$$u = -\eta \,\mathrm{sgn}\,[x_5 (p_4 x_3 - p_3 x_4)] \tag{5-9}$$

when either of the conditions in (5-8) are violated. It is convenient to define the <u>switching function</u>

$$\lambda = x_5 (p_4 x_3 - p_3 x_4)$$

The value of the control u is assumed to be sufficiently large so that only solutions on the boundary (i.e., of the form (5-9)) will be considered.

Simplified Problem - For the simplified system it is assumed that the coefficient k is zero, and that the term, $S\rho V$ is constant, and normalized at unity, i.e.,

$$
\begin{aligned}
x_5 &= 1 \\
x_6 &= 0
\end{aligned}
\tag{5-10}
$$

The state equations for the simplified system are then

$$
\begin{aligned}
\dot{X}_0 &= X_1 X_3 \\
\dot{X}_1 &= X_3 \\
\dot{X}_2 &= X_4 \\
\dot{X}_3 &= -cX_3 - UX_4 \\
\dot{X}_4 &= UX_3 - cX_4 - g
\end{aligned}
\tag{5-11}
$$

where at the terminal time T

$$X_2(T) = 0$$

The Hamiltonian for this system is

$$H = P_0 X_1 X_3 + P_1 X_3 + P_2 X_4 + P_3(-cX_3 - UX_4) + P_4(UX_3 - cX_4 - g) \tag{5-12}$$

The adjoint variables P_ℓ are given by

$$
\begin{aligned}
\dot{P}_0 &= 0 \\
\dot{P}_1 &= -P_0 X_3 \\
\dot{P}_2 &= 0 \\
\dot{P}_3 &= -X_1 P_0 - P_1 + cP_3 - UP_4 \\
\dot{P}_4 &= -P_2 + UP_3 + cP_4
\end{aligned}
\tag{5-13}
$$

with boundary conditions

$$
\begin{aligned}
P_0(T) &= -1 \\
P_1(T) &= P_3(T) = P_4(T) = 0 \\
P_2(T) &= \bar{P}_2
\end{aligned}
$$

The optimum control for the simplified system is

$$U = -\eta \operatorname{sgn} (P_3X_4 - P_4X_3) \tag{5-14}$$

where the switching function Λ is defined by

$$\Lambda = P_3X_4 - P_4X_3 \tag{5-15}$$

Defining $s = \tau - t$ where t is the current time, and τ is the independent time variable, and

$$\alpha = \frac{g}{c^2 + \eta^2}$$

$$A = \frac{cX_3(t) + \tilde{\eta}X_4(t)}{c^2 + \eta^2}$$

$$B = \frac{X_3(t)\tilde{\eta} - cX_4(t) + g}{c^2 + \eta^2}$$

$$\tilde{\eta} = \pm \eta$$

The solution of (5-11) in any subinterval $[\tau, t]$ in which U may be taken as constant is given by

$$\begin{aligned}
X_3(\tau) &= e^{-cs}\{[X_3(t) + \alpha\tilde{\eta}] \cos \tilde{\eta}s - [X_4(t) - \alpha c] \sin \tilde{\eta}s\} - \tilde{\eta}\alpha \\
X_4(\tau) &\doteq e^{-cs}\{[X_3(t) + \alpha\tilde{\eta}] \sin \tilde{\eta}s + [X_4(t) - \alpha c] \cos \tilde{\eta}s\} + c\alpha \\
X_1(\tau) &= e^{-cs}[-A \cos \tilde{\eta}s + B \sin \tilde{\eta}s] - \tilde{\eta}\alpha s - A + X_1(t) \\
X_2(\tau) &= e^{-cs}[A \sin \tilde{\eta}s + B \cos \tilde{\eta}s] + c\alpha s + B + X_2(t)
\end{aligned} \tag{5-16}$$

Consequently, in any sub-interval in which U is constant, the solution is only a function of the state at the beginning of the interval and the length of the interval. At the terminal time, the states $X_1(T)$, $X_2(T)$, $X_3(T)$, $X_4(T)$ depend only on the initial conditions $X_1(t)$, $X_2(t)$, $X_3(t)$, $X_4(t)$, and the number and duration of switching intervals; hence there exists a transformation f such that

$$X(T) = f\,(X(t),\ t_s^1,\ t_s^2,\ \ldots,\ t_s^k,\ T) \tag{5-17}$$

where t_s^{ℓ} is the $\underline{\ell}$th switching time. The solution of the adjoint in a sub-interval $[t, \tau]$ in which U is constant is given by

$$P_0(t) = -1$$

$$P_1(t) = [X_1(t) - X_1(\tau)] + P_1(\tau) \tag{5-18}$$

$$P_2(t) = \bar{P}_2 = \text{constant}$$

$$P_3(t) = e^{cs}\{(P_3(\tau) - A(\tau)) \cos \tilde{\eta} s + (P_4(\tau) - B(\tau) \sin \tilde{\eta} s\} + A(\tau)$$

$$P_4(t) = e^{cs}\{(-P_3(\tau) + A(\tau)) \sin \tilde{\eta} s + (P_4(\tau) - B(\tau)) \cos \tilde{\eta} s\} + B(\tau)$$

where $s = t - \tau$

$$A(\tau) = \frac{c(X_1(\tau) - P_1(\tau)) - \tilde{\eta}\bar{P}_2}{c^2 + \eta^2}$$

$$B(\tau) = \frac{\tilde{\eta}(X_1(\tau) - P_1(\tau)) + c\bar{P}_2}{c^2 + \eta^2}$$

$$\tilde{\eta} = \pm \eta$$

At any time t, the adjoint $P(t)$ is determined by the terminal time T, the i switching times t_s^i between t and T and the value of the adjoint $\bar{P}_2$. Hence, it is possible to define a transformation g such that

$$P(t) = g(X(T), t_s^1, t_s^2, \ldots, t_s^k, T) \tag{5-19}$$

Substituting (5-17) in (5-18) gives the transformation

$$P(t) = G(X(t), t_s^1, t_s^2, \ldots, t_s^k, T) \tag{5-20}$$

A possible algorithm for determining $P(t)$ given $X(t)$ would involve the following steps:

1. Assume that between t and T there are no switching times.
2. Compute T from $X_2(T) = 0$, and compute $X(T)$ from (5-17) and $\bar{P}_2$ from $H(X, P, T) = 0$.
3. Compute $P(t)$ from (5-18).
4. Compute $\Lambda(\tau)$ from (5-15). If $\Lambda(\tau) \neq 0$ for $t < \tau < T$, then $P(t)$ is the required solution.

5. If the switching function $\Lambda(\tau)$ passes through zero, then determine those times t_i $i = 1, \ldots, k$ at which $\Lambda(\tau_i) = 0$ and repeat steps 2 and 3 with these values as switching times.
6. Compute the times τ_i for which $\Lambda(\tau_i') = 0$. If these times τ_i' are the same as τ_i then P(t) is the required solution.
7. If the times τ_i' are different from τ_i then repeat steps 2, 3, and 6 using τ_i' in place of τ_i.

The procedure works if the τ_i' converge to the τ_i.

Quasi-Optimum Control Law - The quasi-optimum controller is given by

$$u = -\eta \,\mathrm{sgn}\,\{x_5(x_3P_4 - x_4P_3) + x_5\xi'[\sum_{i=0}^{6} (x_3M_{i4} - x_4M_{i3})]\} \tag{5-21}$$

M_{i3} and M_{i4} are columns of the matrix Riccati equation (19) and $\xi = x - X$

In this case the coefficient matrices are given by

$$H_{PX} = \begin{bmatrix} 0 & X_3 & 0 & X_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -c & -\tilde{\eta} & -(cX_3 + \tilde{\eta}X_4) & -\eta^2 X_3 \\ 0 & 0 & 0 & \tilde{\eta} & -c & (\tilde{\eta}X_3 - cX_4) & -\eta^2 X_4 \\ 0 & 0 & 0 & \dfrac{2gX_3X_4}{V^4} & \dfrac{2gX_4^2}{V^4} & -(2c + X_4(\dfrac{g}{V^2} + \beta)) & -\eta^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \eta\delta(\Lambda) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_4P_4 & -X_3P_4 & 0 & 0 \\ 0 & 0 & 0 & -X_3P_4 & X_3P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5-22}$$

$$H_{XX} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & (-cP_3 + \tilde{\eta} P_4) & -\eta^2 P_3 \\ 0 & 0 & 0 & 0 & 0 & (-\tilde{\eta} P_3 - cP_4) & -\eta^2 P_4 \\ 0 & 0 & 0 & (-cP_3 + \tilde{\eta} P_4) & (-\tilde{\eta} P_3 - cP_4) & 0 & -\eta^2 (X_3 P_3 + X_4 P_4) \\ 0 & 0 & 0 & -\eta^2 P_3 & -\eta^2 P_4 & -\eta^2 (X_3 P_3 + X_4 P_4) & 0 \end{bmatrix}$$

$$+ \eta \delta(\Lambda) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_4^2 & P_3 P_4 & 0 & 0 \\ 0 & 0 & 0 & P_3 P_4 & -P_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5\text{-}23)$$

$$H_{PP} = \eta \delta(\Lambda) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_4^2 & X_3 X_4 & 0 & 0 \\ 0 & 0 & 0 & X_3 X_4 & -X_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\delta(\Lambda)$ is the delta function and $V = (X_3^2 + X_4^2)^{\frac{1}{2}}$

Because the coefficient matrices of the Riccati equation are time dependent, the auxiliary equations cannot be solved analytically. The components needed for the quasi-optimum controller (5-21) can be determined by either finding an asymptotic solution to (19) or by numerically integrating the equations backwards in time. The boundary conditions for the integration as determined by (27) and (41) are

$$M(T) = -\frac{1}{\dot{X}_2(T)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_3(T) & 0 & 0 \\ 0 & -X_3(T) & Z(T) & -X_1(T) & \bar{P}_2 \\ 0 & 0 & -X_1(T) & 0 & 0 \\ 0 & 0 & \bar{P}_2 & 0 & 0 \end{bmatrix} \tag{5-24}$$

where $Z(T) = -\frac{1}{\dot{X}_2(T)} [\bar{P}_2 \dot{X}_4(T) - \ddot{X}_0(T)]$

In the process of solving for the adjoint $P(t)$, the terms needed to compute $M(T)$ are determined. Hence, a procedure for determining the quasi-optimum correction would consist of first determining the adjoint $P(t)$ given the state $X(t)$, and simultaneously computing the terminal conditions (5-24). The Riccati equation can then be integrated backwards, and the quasi-optimum control determined by (5-21).

2.6 ADAPTIVE CONTROL

Combination of the technique of quasi-optimum control of the mildly nonlinear process $\dot{x} = Ax + \mu f(x) + Bu$ (described in Section 1.7) with the Kalman filtering technique to estimate the small parameter μ results in an "adaptive" control system.

Problem Statement - Consider the problem of minimizing

$$V = x_0(T) = \tfrac{1}{2}\int_{t}^{T} (x'Rx + u'Qu)\, dt \tag{6-1}$$

where T is fixed, Q and R are positive-definite matrices and the process is governed by

$$\dot{x} = Ax + \mu Fx + Bu \tag{6-2}$$

where μ is a small parameter.

This problem has been solved using the quasi-optimum feedback control technique in Section 1.7 to yield the quasi-optimum control law

$$u = Q^{-1}B'(K + \mu L)x \tag{6-3}$$

where

$$-\dot{K} = KA + A'K + KBQ^{-1}B'K - R \tag{6-4}$$

$$K(T, T) = 0 \tag{6-5}$$

and

$$-\dot{L} = L(A + BQ^{-1}B'K) + (A' + KBQ^{-1}B')L + KF + F'K \tag{6-6}$$

Using the above control law the closed loop system becomes

$$\dot{x} = (A + BQ^{-1}B'K)x + \mu(F + BQ^{-1}B'L)x \tag{6-7}$$

which can be represented by the block diagram shown in Figure (6-1).

These results assume, of course, that the state of the process and the parameter μ are known at all times and that μ is a constant. Suppose, however, we do not have complete knowledge of x or μ. Then a sensing device would be used to measure the state and an appropriate filter would construct a best estimate of the state x and the parameter μ. The situation is as shown in Figure 6-2.

Development of Optimum Estimates - If we define

$$C = A + BQ^{-1}B'K \qquad (6\text{-}8)$$

$$D = F + BQ^{-1}B'L \qquad (6\text{-}9)$$

then the process equations are

$$\dot{\mu} = 0 + w_{\mu} \qquad (6\text{-}10)$$

$$\dot{x} = Cx + \mu Dx + w_x \qquad (6\text{-}11)$$

where the observations are

$$y = x + v \qquad (6\text{-}12)$$

Suppose the stochastic process $z(t) = \{w(t), v(t)\}$ is a "white" Gaussian process with correlation function

$$E\{z(t)z'(\tau)\} = \begin{bmatrix} Q(t) & 0 \\ 0 & R_*(t) \end{bmatrix} \delta(t - \tau) \qquad (6\text{-}13)$$

where $\delta(t)$ is a Dirac delta function, and

$$w(t) = \begin{bmatrix} w_{\mu} \\ w_x \end{bmatrix} , \quad Q(t) = \begin{bmatrix} q_{\mu\mu} & q'_{\mu x} \\ q_{\mu x} & Q_{xx} \end{bmatrix} \qquad (6\text{-}14)$$

To obtain the maximum likelihood estimate of $\{\mu(t), x(t)\}$ given $y(t)$, define the following:

$$\tilde{x} = \begin{bmatrix} \mu \\ x \end{bmatrix} , \quad \tilde{y} = \begin{bmatrix} y_{\mu} \\ y \end{bmatrix} \quad \tilde{v} = \begin{bmatrix} 0 \\ v \end{bmatrix} \qquad (6\text{-}15)$$

Then the above process and observation equations become

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 \\ 0 & C + \mu D \end{bmatrix} \tilde{x} + w \qquad (6\text{-}16)$$

$$\dot{\tilde{y}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tilde{v} \qquad (6\text{-}17)$$

The maximum likelihood estimate of $\tilde{x}$ given $\tilde{y}$ can be obtained from the solution to the following two point boundary value problem.

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 \\ 0 & C + \mu D \end{bmatrix} \tilde{x} - \begin{bmatrix} q_{\mu\mu} & q'_{\mu x} \\ q_{\mu x} & Q_{xx} \end{bmatrix} \tilde{p} \tag{6-18}$$

$$\dot{\tilde{p}} = - \begin{bmatrix} 0 & x'D' \\ 0 & C' + \mu D' \end{bmatrix} \tilde{p} + \begin{bmatrix} 0 & 0 \\ 0 & R_*^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y - x \end{bmatrix} \tag{6-19}$$

$$\psi(\tilde{x}(0)) = \tilde{p}(0) \tag{6-20}$$

$$\tilde{p}(t) = 0 \tag{6-21}$$

where

$$\tilde{p} = \begin{bmatrix} P_\mu \\ P_x \end{bmatrix} \tag{6-22}$$

To solve the two point boundary value problem, introduce the vector $\bar{\tilde{x}}$ and the symmetric matrix $P(\tau)$

$$\bar{\tilde{x}}(\tau) = \tilde{x}(\tau) + P(\tau)\tilde{p}(\tau) \tag{6-23}$$

which can be written

$$\begin{bmatrix} \bar{\mu}(\tau) \\ \bar{x}(\tau) \end{bmatrix} = \begin{bmatrix} \mu(\tau) \\ x(\tau) \end{bmatrix} + \begin{bmatrix} P_{\mu\mu} & P'_{\mu x} \\ P_{\mu x} & P_{xx} \end{bmatrix} \begin{bmatrix} P_\mu \\ P_x \end{bmatrix} \tag{6-24}$$

Then

$$\dot{\bar{\tilde{x}}}(\tau) = \dot{\tilde{x}}(\tau) + P(\tau)\dot{\tilde{p}}(\tau) + \dot{P}(\tau)\tilde{p}(\tau) \tag{6-25}$$

and

$$\begin{aligned} \dot{\bar{\tilde{x}}}(\tau) = & \begin{bmatrix} 0 & 0 \\ 0 & C + \mu D \end{bmatrix} \bar{\tilde{x}} - \begin{bmatrix} 0 & 0 \\ Dx & C + \mu D \end{bmatrix} P\tilde{p} - \begin{bmatrix} q_{\mu\mu} & q'_{\mu x} \\ q_{\mu x} & Q_{xx} \end{bmatrix} \tilde{p} \\ & - P \begin{bmatrix} 0 & x'D' \\ 0 & C' + \mu D' \end{bmatrix} \tilde{p} + P \begin{bmatrix} 0 & 0 \\ 0 & R_*^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y - \bar{x} \end{bmatrix} + P \begin{bmatrix} 0 & 0 \\ 0 & R_*^{-1} \end{bmatrix} Pp + \dot{P}p \end{aligned} \tag{6-26}$$

or, with

$$\dot{P} = \begin{bmatrix} 0 & 0 \\ D_x & C + \mu D \end{bmatrix} P + P \begin{bmatrix} 0 & x'D' \\ 0 & C' + \mu D' \end{bmatrix} + \begin{bmatrix} q_{\mu\mu} & q'_{\mu x} \\ q_{\mu x} & Q_{xx} \end{bmatrix} - P \begin{bmatrix} 0 & 0 \\ 0 & R_*^{-1} \end{bmatrix} P \tag{6-27}$$

we have

$$\dot{\bar{\tilde{x}}} = \begin{bmatrix} 0 & 0 \\ 0 & C + \mu D \end{bmatrix} \bar{\tilde{x}} + P \begin{bmatrix} 0 & 0 \\ 0 & R_*^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y - \bar{x} \end{bmatrix} \tag{6-28}$$

In expanded form, equations (6-27) and (6-28) become

$$\dot{\hat{\mu}} = P'_{\mu x} R_*^{-1} (y - \hat{x}) \tag{6-29}$$

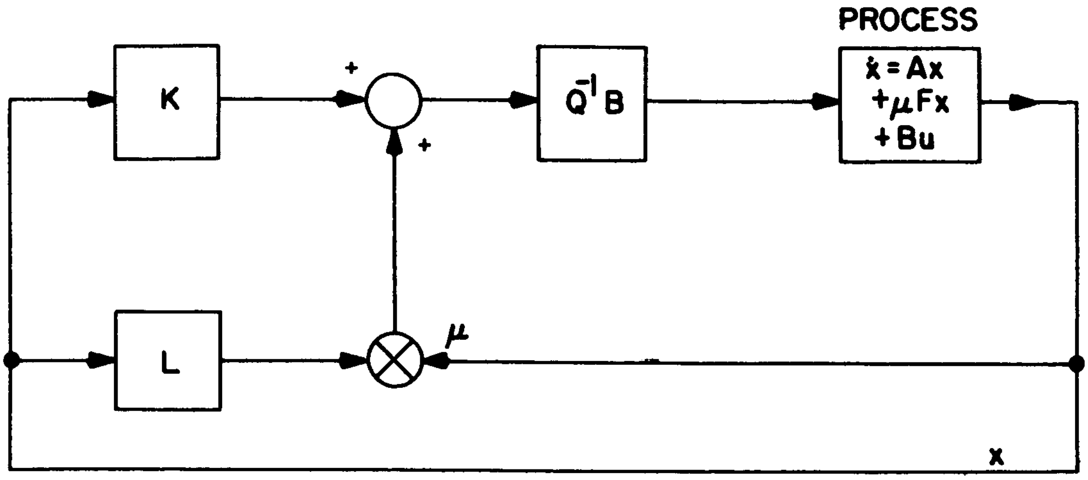
$$\dot{\hat{x}} = [C + \hat{\mu} D] \hat{x} + P_{xx} R_*^{-1} (y - \hat{x}) \tag{6-30}$$

$$\dot{P}_{\mu\mu} = q_{\mu\mu} - P'_{\mu x} R_*^{-1} P_{\mu x} \tag{6-31}$$

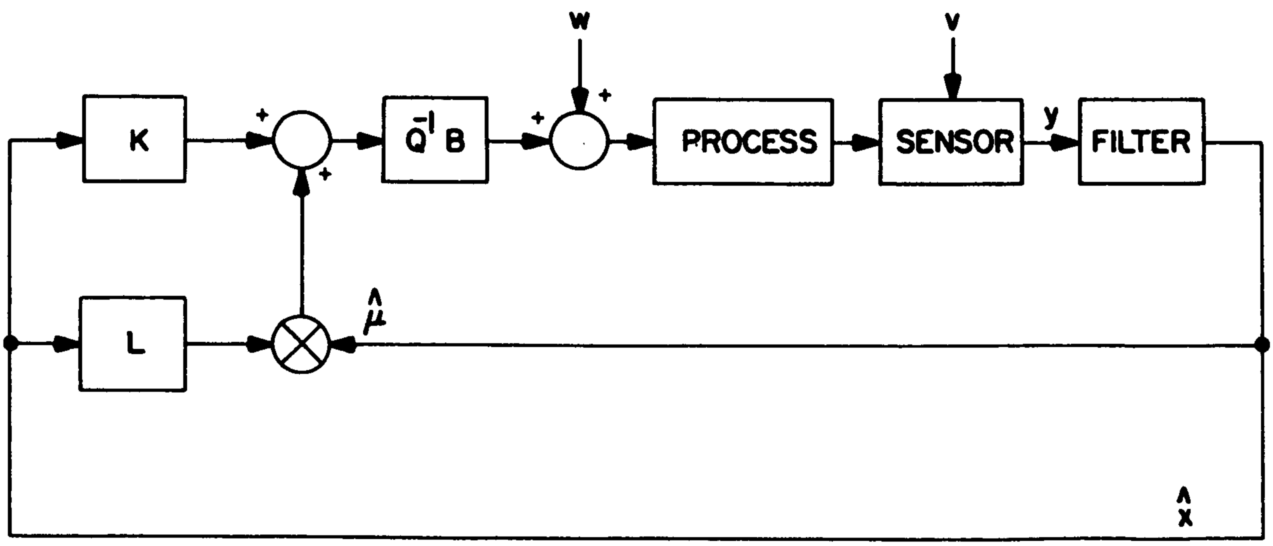
$$\dot{P}_{\mu x} = (C + \hat{\mu} D) P_{\mu x} - P_{xx} R^{-1} P_{\mu x} + q_{\mu x} + P_{\mu\mu} D\hat{x} \tag{6-32}$$

$$\dot{P}_{xx} = (C + \hat{\mu} D) P_{xx} + P_{xx}(C' + \hat{\mu} D') - P_{xx} R_*^{-1} P_{xx} + Q_{xx} + D\hat{x} P'_{\mu x} + P_{\mu x} \hat{x}' D' \tag{6-33}$$

where (6-29) thru (6-33) give the optimum estimates to be used in the system of Figure 6-2.



QUASI-OPTIMUM CONTROL SYSTEM
FIGURE 6-1



ADAPTIVE CONTROL SYSTEM
FIGURE 6-2

CONCLUSIONS AND RECOMMENDATIONS

On the basis of the results achieved in the examples considered, it is our conclusion that the quasi-optimum control technique described herein is a valuable tool for the design of practical feedback control systems. Like all engineering methods, it is not a panacea, and there are, no doubt, many situations for which other methods are more suitable. In order for our method to be applicable to a particular design problem, two conditions must be met. First, the actual process must be capable of being approximated by a simpler process, and, second, the exact control law for the simpler process must be found. Experience with the physical problem to be solved is an aid to meeting the first requirement, and familiarity with the solved problems of optimum control is an aid to meeting the second. The successful application of the technique to a particular design problem, however, will ultimately depend on the user's ingenuity. We regard this as an asset, not a shortcoming of the technique.

Although the correction matrix M can be expressed in terms of the fundamental (transition) matrix of a linear system or as the solution to a matrix Riccati equation, the analytical determination of the matrix is at best an extremely tedious chore and may be impossible. As a consequence, either numerical integration methods or additional approximations will be required to obtain M. Since the numerical integration is performed off-line and is feasible even for systems as high as 20th order (i.e., 210 simultaneous ordinary equations)— perhaps even higher —this is a factor in the cost of using the method but, in our opinion, is not a serious limitation.

A major unresolved theoretical question entails the performance of the quasi-optimum control law. Suppose that the performance criterion to be minimized by the optimum feedback control law is given by V^0. The performance V^q achieved by use of the quasi-optimum control law, while greater than V^0 should certainly be smaller than the performance V^s achieved by using the control law for the simplified process. Thus it should be possible to demonstrate that $V^0 < V^q \leq V^s$ for a sufficiently small range of ξ, which will be the range of validity of the technique. The demonstration of this, and the estimation of the difference $\Delta V = V^q - V^0$ should be investigated further. Since the quality of performance of the quasi-optimum control law in any practical instance will almost certainly be established by computer simulation, however, the lack of knowledge of the range of validity of the technique is not a handicap, provided the user is optimistic enough to try it out. The limited number of examples considered in the study lead us to conclude that an optimistic viewpoint is justified.

The stability of the quasi-optimum control law is another problem of major importance which should receive attention. This problem can probably be approached by the same techniques used to establish the performance of the quasi-optimum control law.

From the theoretical standpoint, the other uses of the quasi-optimum control technique considered in Section 1.5 should receive more attention. In this regard it is also worth considering whether the analysis can also be used to develop a trajectory optimization technique which would be a generalization and a simplification of the technique described by McReynolds and Bryson [9].

More completed examples will add practical insight into the advantages and limitations of the technique. Consequently we recommend that the studies of multiple-axis attitude control, and reentry guidance, described in Sections 2.4 and 2.5, respectively, be completed, and the studies of the application of the method to other problems in guidance and control be undertaken.

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